SEMIHOMOMORPHISMS OF SEMIMODULAR LATTICES¹

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In this paper a class of lattice "seminomomorphisms" is defined in a manner analogous to that of lattice semimodularity. It is shown that a seminomomorphism of a modular lattice is a homomorphism; while on a semimodular lattice which satisfies a certain "regularity" property, seminomomorphisms are uniquely determined by specifying their kernels. The definitions and results are given below for the lower semimodular case; their duals for upper semimodularity are straight forward. With the exception of Theorems 3 and 12, all results extend immediately to the case where $\mathcal{L}$ is a finite-dimensional lower semilattice, and this fact is used in the proof of Theorem 12.

For any element $x \neq 0$ of a lattice $\mathcal{L}$, let A_x denote the set of atoms $a \in \mathcal{L}$ (elements which cover 0) such that $a \leq x$; in particular $A_0 = \emptyset$. The cardinality of this set, $|A_x|$, we will call the *A-rank* of x . A finite dimensional lattice in which $|A_x| = \dim x$ for all x will be called *A-regular*. It is easy to show that in any lattice $x \leq y$ implies $A_x \subseteq A_y$; $A_{x \vee y} \supseteq A_x \cup A_y$; and $A_{x \wedge y} = A_x \cap A_y$. Further, if $\mathcal{L}$ is A-regular then $x = \sup\{A_x\}$, and consequently $A_x = A_y$ if and only if $x = y$, and $A_x \subset A_y$ if and only if $x < y$.

Let $S = \{a_1, \dots, a_n\}$ be any set of atoms in $\mathcal{L}$. We call S *full* if $A_{\sup S} = S$. It follows that in an A-regular lattice, S is full if and only if $S = A_x$ for some $x \in \mathcal{L}$. As an immediate consequence we have: if $\mathcal{L}$ is A-regular then $\{(x, A_x) \mid x \in \mathcal{L}\}$ is an order isomorphism of $\mathcal{L}$ with the set of full sets of atoms of $\mathcal{L}$ ordered by inclusion. It can also be shown that if $\mathcal{L}$ is A-regular then an element x covers at least two distinct elements if and only if $\dim x \geq 2$.

A lattice is said to be *lower semimodular* (LSM) if $x \vee y$ covers both x and y , then x and y both cover $x \wedge y$. It is well known that semimodular lattices satisfy the Jordan-Dedekind condition and that dimension is additive [1]. A-regular, lower semimodular lattices will be called *G-lattices*. G-lattices arise naturally in the study of directed graphs; if one partially orders by inclusion the set of convex subgraphs of any finite directed graph, then the resulting lattice is lower semimodular and A-regular [2]. G-lattices have several interesting structural properties, but we shall need only the result given below.

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PROPOSITION 1. *Let x cover the distinct elements y_1, y_2, y_3 in the G -lattice $\mathcal{L}$, and let $z_1 = y_2 \wedge y_3, z_2 = y_1 \wedge y_3, z_3 = y_1 \wedge y_2$; then $z_1 \wedge z_2 = z_1 \wedge z_3 = z_2 \wedge z_3$.*

Proof. Readily $\dim x = k \geq 3$, and since x covers the y 's, by the Jordan-Dedekind condition they each have dimension $k-1$; while by LSM, the y 's cover the z 's, so that each z has dimension $k-2$.

If $z_1 = z_2$, we would have $z_1 < y_1, y_2, y_3$, so that $A_{y_i} = A_{z_1} \cup \{a_i\}$, $i = 1, 2, 3$, where the a_i 's are distinct since the y_i 's are distinct. Hence $|A_x| \geq |A_{z_1} \cup \{a_1, a_2, a_3\}| = k+1$, contradiction; we have thus shown that the z 's are all distinct.

Suppose that $z_1 \wedge z_2 \neq z_1 \wedge z_3$. Since y_3 covers both z_1 and z_2 , by LSM z_1 covers $z_1 \wedge z_2$. Similarly, since y_2 covers z_1 and z_3 , z_1 covers $z_1 \wedge z_3$. Thus $z_1 \wedge z_2 \neq z_1 \wedge z_3$ implies $(z_1 \wedge z_2) \vee (z_1 \wedge z_3) = z_1$. Moreover, $z_1 \wedge z_2 \leq z_2 < y_1$ and $z_1 \wedge z_3 \leq z_3 < y_1$, so that their sup is $\leq y_1$. Thus $z_1 = y_2 \wedge y_3 \leq y_1$, so that $y_2 \wedge y_3 \leq y_1 \wedge y_3$, i.e., $z_1 \leq z_2$, and since they have the same dimension, this implies that $z_1 = z_2$, contradiction. $\square$

We define a *lower semihomomorphism* (LSH) to be a mapping $\sigma: \mathcal{L} \rightarrow \mathcal{L}'$ such that, for all x, y in $\mathcal{L}$,

- (a) $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$,
- (b) if $x \vee y$ covers x and y then, $\sigma(x \vee y) = \sigma(x) \vee \sigma(y)$.

[Note the analogy to the definition of LSM.]

We first state some needed results about meet homomorphisms. For brevity, we will denote $\inf\{y \in \mathcal{L} \mid y \in \sigma^{-1}(\sigma(x))\}$ by $\hat{x}$, or by $(x)^\wedge$ when x is an involved expression.

PROPOSITION 2. *If σ is a meet homomorphism defined on $\mathcal{L}$, where $\mathcal{L}$ is finite dimensional, then*

- (1) $\sigma(0)$ is the zero element of $\sigma(\mathcal{L})$.
- (2) If $x \vee y$ exists in $\mathcal{L}$ then $\sigma(x) = \sigma(x \vee y)$ implies $\sigma(y) = \sigma(x \wedge y)$.
- (3) $\sigma(\hat{x}) = \sigma(x)$.
- (4) If $y' < \sigma(x)$ in $\sigma(\mathcal{L})$ then there exists $y \in \sigma^{-1}(y')$ such that $y < x$ in $\mathcal{L}$.
- (5) $\sigma(y) < \sigma(x)$ in $\sigma(\mathcal{L})$ implies $\hat{y} < \hat{x}$.
- (6) If $\hat{x} \vee \hat{y}$ exists in $\mathcal{L}$ then $\sigma(x) \vee \sigma(y)$ exists in $\sigma(\mathcal{L})$ and equals $\sigma(\hat{x} \vee \hat{y})$. In particular if $\mathcal{L}$ is a lattice, so is $\sigma(\mathcal{L})$.

We now prove a result which can be regarded as justifying our definition of a lower semihomomorphism.

THEOREM 3. *If $\sigma: \mathcal{L} \rightarrow \mathcal{L}'$ is an LSH and $\mathcal{L}$ is a modular lattice, then σ is a lattice homomorphism.*

PROOF. We must show that $\sigma(x \vee y) = \sigma(x) \vee \sigma(y)$ for all x, y in $\mathcal{L}$.

We proceed by induction on $k = \dim(x \vee y : x \wedge y)$. If $k = 1$, we must have either $x = x \vee y$, $y = x \wedge y$ or vice versa, say the former, so that $y < x$, and we have $\sigma(x \vee y) = \sigma(x) = \sigma(x) \vee \sigma(y)$ since $\sigma(y) \leq \sigma(x)$. Hence we may assume that $\sigma(u \vee v) = \sigma(u) \vee \sigma(v)$ for all u, v such that $\dim(u \vee v : u \wedge v) < k$. As in the case $k = 1$, if $x \leq y$ or $y \leq x$ the result is trivial; hence we may suppose $x < x \vee y$ and $y < x \vee y$. Let $x \leq x_1 < x \vee y$, $y \leq y_1 < x \vee y$, where $x \vee y$ covers x_1 and y_1 . Since $\mathcal{L}$ is modular and $x \leq x_1$, we have $(x_1 \wedge y) \vee x = x_1 \wedge (x \vee y) = x_1$, and similarly $(x \wedge y_1) \vee y = y_1$. Thus $\dim((x_1 \wedge y) \vee x : (x_1 \wedge y) \wedge x) = \dim(x_1 : x \wedge y) = k - 1$ since $x \vee y$ covers x_1 , so that by induction hypothesis we have $\sigma(x_1) = \sigma((x_1 \wedge y) \vee x) = \sigma(x_1 \wedge y) \vee \sigma(x)$, and similarly $\sigma(y_1) = \sigma(x \wedge y_1) \vee \sigma(y)$. Finally, since $x \vee y$ covers x_1 and y_1 , and σ is an LSH, we have $\sigma(x \vee y) = \sigma(x_1 \vee y_1) = \sigma(x_1) \vee \sigma(y_1)$. But as just shown, this is in turn equal to $(\sigma(x_1 \wedge y) \vee \sigma(x)) \vee (\sigma(x \wedge y_1) \vee \sigma(y)) = (\sigma(x) \vee \sigma(x \wedge y_1)) \vee (\sigma(y) \vee \sigma(x \wedge y)) = \sigma(x) \vee \sigma(y)$. $\square$

PROPOSITION 4. *Let $\sigma : \mathcal{L} \rightarrow \mathcal{L}'$ be an LSH, and let $\mathcal{L}$ be LSM. Then $\dim(y : x) \leq 1$ implies $\dim(\sigma(y) : \sigma(x)) \leq 1$ —in other words, if y covers x , we have either $\sigma(y) = \sigma(x)$, or $\sigma(y)$ covers $\sigma(x)$ in $\sigma(\mathcal{L})$.*

PROOF. Let $\sigma(x) \leq z' \leq \sigma(y)$; we must show that $z' = \sigma(x)$ or $\sigma(y)$. Recall first that by Proposition 2 we have $\hat{x} \leq \hat{z} \leq y$. We proceed by induction on $k = \dim(x : \hat{x})$. If $k = 0$, then $x = \hat{x}$ and we have $x = \hat{x} \leq \hat{z} \leq y$. Since y covers x , this implies that either $\hat{z} = x$, so that $z' = \sigma(x)$, or else $\hat{z} = y$, so that $z' = \sigma(y)$. Hence we may assume that if $\dim(u : v) = 1$ and $\dim(v : \hat{v}) < k$, then $\dim(\sigma(u) : \sigma(v)) \leq 1$. If $\hat{z} = y$ we have $z' = \sigma(\hat{z}) = \sigma(y)$; if not, let $\hat{z} \leq z_1 < y$, where y covers z_1 . If $x = z_1$, we have $\hat{z} \leq x$, i.e. $\hat{z} = \hat{x}$, and we are done. Thus we may assume that $x \neq z_1$. Since y covers both x and z_1 , it is their sup, and since σ is an LSH, we thus have $\sigma(y) = \sigma(x) \vee \sigma(z_1)$; but $\sigma(x) \leq z' = \sigma(\hat{z}) \leq \sigma(z_1)$, so that $\sigma(y) = \sigma(z_1)$. Moreover, by LSM, x and z_1 both cover $x \wedge z_1$, while $\hat{x} \leq z \leq z_1$ and $\hat{x} \leq x$ imply $\hat{x} \leq x \wedge z_1$. Hence by additivity of dimension we have $\dim((x \wedge z_1) : \hat{x}) = k - 1$. Furthermore, $\hat{x} \leq x \wedge z_1 \leq x$ implies $\sigma(x \wedge z_1) = \sigma(x)$, so that $(x \wedge z_1)^\wedge = \hat{x}$. We can now apply the induction hypothesis with $u = z_1$, $v = x \wedge z_1$, to obtain $\dim(\sigma(z_1) : \sigma(x \wedge z_1)) = \dim(\sigma(y) : \sigma(x)) \leq 1$. $\square$

THEOREM 5. *LSH's take LSM lattices into LSM lattices.*

PROOF. Let x', y' in $\sigma(\mathcal{L})$ be such that $x' \vee y'$ covers both x' and y' , and let $\hat{z} = \inf \{ \sigma^{-1}(x' \vee y') \}$. By Proposition 2, we have $\hat{x}$ and $\hat{y} \leq \hat{z}$. Let $\hat{x} \leq x_1 < \hat{z}$ and $\hat{y} \leq y_1 < \hat{z}$, where z covers both x_1 and y_1 . Now $x' = \sigma(\hat{x}) \leq \sigma(x_1) \leq \sigma(\hat{z}) = x' \vee y'$. Since $x' \vee y'$ covers x' , and $\sigma(x_1) \neq x' \vee y'$ by definition of $\hat{z}$, we must have $\sigma(x_1) = x'$, and similarly $\sigma(y_1) = y'$.

Since $\mathfrak{L}$ is LSM, x_1 and y_1 both cover $x_1 \wedge y_1$. Also $\sigma(x_1 \wedge y_1) = \sigma(x_1) \wedge \sigma(y_1) = x' \wedge y'$. Thus (using Proposition 4) we have $1 \leq \dim(x' : x' \wedge y')$ (since $x' \neq x' \wedge y'$) $= \dim(\sigma(x_1) : \sigma(x_1 \wedge y_1)) \leq \dim(x_1 : x_1 \wedge y_1) = 1$. Consequently $\dim(x' : x' \wedge y') = 1$, and similarly y' covers $x' \wedge y'$, so that $\sigma(\mathfrak{L})$ is LSM. $\square$

Having shown that $\sigma(\mathfrak{L})$ is LSM and hence Jordan-Dedekind, an easy argument on unrefinable chains shows that if σ is a LSH and $\mathfrak{L}$ is LSM then $\dim(\sigma(y) : \sigma(x)) \leq \dim(y : x)$.

Now let $\mathfrak{L}$ be A-regular and σ an LSH. Using the fact that $\dim x \geq 2$ implies that x covers at least two distinct elements one can show that if a' is an atom of $\sigma(\mathfrak{L})$ then there exists exactly one atom $a \in \mathfrak{L}$ such that $\sigma(a) = a'$. It follows that for any $x \in \mathfrak{L}$, we have $A_{\sigma(x)} = \sigma(A_x)$ (after deleting any zeros in $\sigma(A_x)$).

THEOREM 6. *LSH's take G-lattices into G-lattices.*

PROOF. By Theorem 5, it suffices to prove A-regularity—in other words, that $|A_{x'}| = \dim x'$ for all $x' \in \sigma(\mathfrak{L})$. We proceed by induction on $k = \dim x'$. If $k = 1$, x' is an atom and the conclusion is immediate; suppose it true for dimensions $< k$, where $k \geq 2$.

Since x' is not an atom, neither is $\hat{x}$ (Proposition 4), so that $\hat{x}$ covers at least two elements y_1, y_2 . Hence x' covers $\sigma(y_1)$ (Proposition 4; we cannot have $\sigma(y_1) = x'$ by definition of $\hat{x}$), and similarly x' covers $\sigma(y_2)$. Furthermore, $\sigma(y_1) \neq \sigma(y_2)$, since otherwise we would have $\sigma(\hat{x}) = \sigma(y_1 \vee y_2) = \sigma(y_1) \vee \sigma(y_2) = \sigma(y_1) \neq x'$.

Since $\sigma(\mathfrak{L})$ is LSM, dimension is additive, so that $\dim \sigma(y_1) = \dim \sigma(y_2) = k - 1$, and by induction hypothesis, $|A_{\sigma(y_1)}| = |A_{\sigma(y_2)}| = k - 1$. Moreover, $A_{\sigma(y_1)} \neq A_{\sigma(y_2)}$, since otherwise we would have $|A_{\sigma(y_1) \wedge \sigma(y_2)}| = |A_{\sigma(y_1)}| = k - 1$; but since $\sigma(\mathfrak{L})$ is LSM, $\sigma(y_1)$ and $\sigma(y_2)$ cover $\sigma(y_1) \wedge \sigma(y_2)$, so that $\dim \sigma(y_1) \wedge \sigma(y_2) = k - 2$, and by induction hypothesis $|A_{\sigma(y_1) \wedge \sigma(y_2)}| = k - 2$, contradiction. Since $A_{x'} \supseteq A_{\sigma(y_1)} \cup A_{\sigma(y_2)}$, we must have $|A_{x'}| \geq k$. On the other hand, let $A_{\hat{x}} = A_{y_1} \cup \{a\}$; then by the preceding comment $A_{x'} = \sigma(A_{y_1}) \cup \{\sigma(a)\} = A_{\sigma(y_1)} \cup \{\sigma(a)\}$, so that $|A_{x'}| \leq (k - 1) + 1 = k$. $\square$

Our goal in the remainder of this paper is to prove that an LSH of a G-lattice is “uniquely determined” by specifying its kernel (= the set of elements which it takes into $0'$). Specifically, we show that if σ and τ have the same kernel, they induce the same equivalence relation on $\mathfrak{L}$, so that their images are canonically order isomorphic;² and that there exists an LSH having any given kernel. In what follows, we denote the kernel of σ by $k(\sigma)$.

² The analogous result for lattice homomorphisms is known to be true for relatively complemented lattices [1, p. 23].

THEOREM 7. *If $\mathcal{L}$ is a G-lattice, and σ and τ are LSH's defined on $\mathcal{L}$ such that $k(\sigma) = k(\tau)$, then $\{(\sigma(x), \tau(x)) \mid x \in \mathcal{L}\}$ is an order isomorphism of $\sigma(\mathcal{L})$ with $\tau(\mathcal{L})$.*

PROOF. Suppose that $\sigma(y) = \sigma(x)$. Then $A_{\sigma(x)} = A_{\sigma(y)}$, so that by remarks preceding Theorem 6, $\sigma(A_x) = \sigma(A_y)$, discarding 0's, and the set A of atoms in A_x which σ does not take into $0'$ is the same as the set of atoms in A_y which σ does not take into $0'$. Since $k(\sigma) = k(\tau)$, it follows that $A_{\tau(x)} = \tau(A_x)$ (discarding 0's) $= \tau(A) = \tau(A_y)$ (discarding 0's) $= A_{\tau(y)}$, so that $\tau(x) = \tau(y)$ (since $\tau(\mathcal{L})$ is A-regular), and we have proved that $\{(\sigma(x), \tau(x)) \mid x \in \mathcal{L}\}$ is a one-to-one correspondence. Finally, if $\sigma(z) \leq \sigma(x)$ we have $\hat{z} \leq \hat{x}$, so that $\tau(\hat{z}) \leq \tau(\hat{x})$, and since $\sigma(\hat{z}) = \sigma(z)$ and $\sigma(\hat{x}) = \sigma(x)$ we have $\tau(\hat{z}) = \tau(z)$, $\tau(\hat{x}) = \tau(x)$, and symmetrically so that the one-to-one correspondence is order preserving both ways. $\square$

COROLLARY 8. *An LSH of a G-lattice is one-to-one if and only if its kernel is $\{0\}$.*

PROOF. If $k(\sigma) = \{0\}$, we can apply Theorem 7 to σ and the identity function on $\mathcal{L}$ to prove that $\{(x, \sigma(x)) \mid x \in \mathcal{L}\}$ is a one-to-one correspondence. $\square$

COROLLARY 9. *A one-to-one LSH of a G-lattice is an order isomorphism.*

Our final goal is to prove that on a G-lattice, there exists an LSH having any given kernel. In order to state this more precisely, we first prove

PROPOSITION 10. *Let σ be an LSH defined on the G-lattice $\mathcal{L}$; then $x \in k(\sigma)$ if and only if $A_x \subseteq k(\sigma)$.*

PROOF. "Only if" is trivial since $k(\sigma)$ is an ideal. "If" is clear if $x = 0$ or is an atom; we proceed by induction on $\dim x = k \geq 2$. Let x cover x_1, x_2 (distinct); then $A_{x_1}, A_{x_2} \subseteq A_x \subseteq k(\sigma)$, while $\dim x_1 = \dim x_2 = k - 1$, so that by induction hypothesis, x_1 and x_2 are in $k(\sigma)$. But since $x = x_1 \vee x_2$, we thus have $\sigma(x) = \sigma(x_1) \vee \sigma(x_2) = 0' \vee 0' = 0'$, proving $x \in k(\sigma)$. $\square$

COROLLARY 11. *Let σ be an LSH defined on a G-lattice, and let A be the set of all atoms in $k(\sigma)$; then $k(\sigma) = \{\sup B \mid B \subseteq A, \text{ where } B \text{ is full}\}$.*

Thus the kernel of an LSH of a G-lattice is completely determined by specifying the set of atoms which it contains. In what follows, we

denote $\{\sup B \mid B \subseteq A, B \text{ full}\}$ by $\mathcal{I}(A)$, since this set is an ideal of the lattice. We can now state

THEOREM 12. *Let A be any set of atoms of the G -lattice $\mathcal{L}$; then there exists an LSH defined on $\mathcal{L}$ whose kernel is $\mathcal{I}(A)$.*

Note that by Corollary 11; this is equivalent to proving that there exists a σ such that A is precisely the set of atoms in $k(\sigma)$.

PROOF. This is trivial if $\dim 1 = 0$ or 1 , since we can take σ to be the trivial LSH ($\sigma(0) = \sigma(1) = 0'$), if $A = \{1\}$, or the identity mapping if $A = \emptyset$. We proceed by induction on $\dim 1 = k \geq 2$. Let 1 cover $z_1, \dots, z_m$ (where $m \geq 2$); since $\dim z_i = k - 1$ for each i , there exist LSH's $\sigma_1, \dots, \sigma_m$ on $\mathcal{L}_{z_1}, \dots, \mathcal{L}_{z_m}$, respectively, such that $k(\sigma_i) = \mathcal{I}(A_{z_i} \cap A)$, $1 \leq i \leq m$, where $\mathcal{L}_x$ denotes the ideal $\{y \mid y \leq x\}$.

For any $x \in \mathcal{L}^* = \mathcal{L} \sim \{1\}$ we have $x \leq z_i$ for some i . Let

$$\hat{x}_i = \inf\{y \in \mathcal{L}^* \mid \sigma_i(y) = \sigma_i(x)\}.$$

Suppose that x also $\leq z_j$; we show that $\hat{x}_i = \hat{x}_j$. In fact, σ_i and σ_j restricted to $\mathcal{L}_x$ have the same kernel, so that in $\mathcal{L}_x$, $\sigma_i(y) = \sigma_i(z)$ if and only if $\sigma_j(y) = \sigma_j(z)$. In particular,

$$\{y \in \mathcal{L}_x \mid \sigma_i(y) = \sigma_i(x)\} = \{y \in \mathcal{L}_x \mid \sigma_j(y) = \sigma_j(x)\},$$

so that the infs of these sets are equal, and since $\hat{x}_i$ and $\hat{x}_j$ are evidently in $\mathcal{L}_x$, they must be just these infs. From now on we denote $\hat{x}_i$ (for every i such that $x \leq z_i$) by $\hat{x}$. Since $\hat{x} = \hat{x}_i$ is the inf of a finite set of y 's for each of which $\sigma_i(y) = \sigma_i(x)$, we have $\sigma_i(\hat{x}) = \sigma_i(x)$ for any i such that $x \leq z_i$.

Let Σ be the equivalence relation $\{(x, y) \mid \hat{x} = \hat{y}\}$, and let σ be the canonical function of Σ . Define a partial order relation on the quotient set $\sigma(\mathcal{L}^*)$ by $\sigma(x) \leq \sigma(y)$ if and only if $\hat{x} \leq \hat{y}$. Let $x \leq y \leq z_i$ (say); since σ_i is a meet homomorphism, $\sigma_i(x) \leq \sigma_i(y)$, so that by (v) of Proposition 2 we have $\hat{x}_i \leq \hat{y}_i$, i.e., $\hat{x} \leq \hat{y}$, proving that σ is order preserving. To show that it is an LSH, we need some preliminary results.

(1) $\hat{x} = \hat{y}$ if and only if $\sigma_i(x) = \sigma_i(y)$ for some i . To see "only if," note that $\sigma_i(x) = \sigma_i(\hat{x}) = \sigma_i(\hat{y}) = \sigma_i(y)$ since $\hat{x} = \hat{y}$. Conversely, by definition of $\hat{x} = \hat{x}_i$, if $\sigma_i(\hat{y}) = \sigma_i(x) = \sigma_i(\hat{x})$ we have $\hat{x} \leq \hat{y}$. Now for some j we have $\sigma_j(\hat{y}) = \sigma_j(y)$, and since σ_i and σ_j restricted to $\mathcal{L}_{\hat{y}}$ have the same kernel, $\sigma_i(\hat{x}) = \sigma_i(\hat{y})$ implies $\sigma_j(\hat{x}) = \sigma_j(\hat{y}) = \sigma_j(y)$, so that by definition of $\hat{y} = \hat{y}_j$ we have $\hat{y} \leq \hat{x}$.

(2) It follows that $(\hat{w})^\wedge = \hat{w}$ for all $w \in \mathcal{L}^*$ —for, take $x = \hat{w}$, $y = w$ in (1). [In terms of σ , this can be restated as: $\sigma(\hat{w}) = \sigma(w)$.]

(3) For any x, y in $\mathcal{L}^*$ we have $(\hat{x} \wedge \hat{y})^\wedge = (\hat{x} \wedge \hat{y})^\wedge$.

PROOF. Let $\sigma_i(\hat{x}) = \sigma_i(x)$; then since $\hat{x} \leq \hat{x} \vee (x \wedge y) \leq x$, we have $\sigma_i(\hat{x}) = \sigma_i(\hat{x} \vee (x \wedge y))$, so that by (ii) of Proposition 2, $\sigma_i(x \wedge y) = \sigma_i(\hat{x} \wedge (x \wedge y)) = \sigma_i(\hat{x} \wedge y)$. Similarly, for some j we have $\sigma_j(\hat{x} \wedge y) = \sigma_j(\hat{x} \wedge \hat{y})$, and (working in $\mathfrak{L}_{x \wedge y}$) this implies $\sigma_i(\hat{x} \wedge y) = \sigma_i(\hat{x} \wedge \hat{y})$. Since $\sigma_i(x \wedge y)^\wedge = \sigma_i(x \wedge y)$, we have thus shown that $\sigma_i(\hat{x} \wedge \hat{y}) = \sigma_i(x \wedge y)^\wedge$; hence by (1) (using $\hat{x} \wedge \hat{y}$ for x , and $x \wedge y$ for y) our conclusion follows. In terms of σ , this conclusion becomes $\sigma(x \wedge y) = \sigma(\hat{x} \wedge \hat{y})$; and by (2), this implies $\sigma(x \wedge y)^\wedge = \sigma(\hat{x} \wedge \hat{y})$. Note also that $(x \wedge y)^\wedge = (\hat{x} \wedge \hat{y})^\wedge$ implies $(x \wedge y)^\wedge \leq \hat{x} \wedge \hat{y}$.

We can now resume the proof that σ is an LSH. Since $(x \wedge y)^\wedge \leq \hat{x} \wedge \hat{y} \leq \hat{x}$, we have $\sigma(x \wedge y) = \sigma(x \wedge y)^\wedge \leq \sigma(\hat{x}) = \sigma(x)$, and similarly $\sigma(x \wedge y) \leq \sigma(y)$. Suppose that $\sigma(z) \leq \sigma(x)$ and $\sigma(y)$, i.e., that $\hat{z} \leq \hat{x}$ and $\hat{y}$, which implies $\hat{z} \leq \hat{x} \wedge \hat{y}$. Then $\hat{z} = (\hat{z})^\wedge \leq (\hat{x} \wedge \hat{y})^\wedge = (x \wedge y)^\wedge$, i.e. $\sigma(z) \leq \sigma(x \wedge y)$, so that $\sigma(x \wedge y)$ is the inf of $\sigma(x)$ and $\sigma(y)$, proving that σ is a meet homomorphism.

Let $x \vee y$ cover x and y in $\mathfrak{L}^*$; since σ is a meet homomorphism, by Proposition 2(vi), $\sigma(x) \vee \sigma(y)$ exists and $= \sigma(\hat{x} \vee \hat{y})$. Let $x \vee y \leq z_k$; since σ_k is an LSH we have $\sigma_k(x \vee y)^\wedge = \sigma_k(x \vee y) = \sigma_k(x) \vee \sigma_k(y) = \sigma_k(\hat{x}) \vee \sigma_k(\hat{y}) = \sigma_k(\hat{x} \vee \hat{y})$ (using Proposition 2(vi)). Hence by (1), $(\hat{x} \vee \hat{y})^\wedge = (x \vee y)^\wedge$, i.e. $\sigma(\hat{x} \vee \hat{y}) = \sigma(x \vee y)$, so that σ is an LSH on $\mathfrak{L}^*$.

It remains to extend σ from $\mathfrak{L}^*$ to $\mathfrak{L}$ by defining it for 1. To this end, note first that the maximal elements of $\sigma(\mathfrak{L}^*)$ must be $\sigma(z_i)$'s—for, if $\sigma(u)$ is maximal, let $u \leq z_j$; then $\sigma(u) \leq \sigma(z_j)$, implying $\sigma(u) = \sigma(z_j)$.

Suppose that $\sigma(z_i) = \sigma(z_j)$, where $i \neq j$; we shall show that in this case $\sigma(z_i)$ is the unique maximal element, i.e. the greatest element of $\sigma(\mathfrak{L}^*)$. Let k be different from i and j (if there are only two z_i 's, the conclusion is immediate). By Proposition 1, since 1 covers z_i, z_j, z_k in $\mathfrak{L}$, then $(z_i \wedge z_j) \wedge (z_i \wedge z_k) = (z_i \wedge z_j) \wedge (z_j \wedge z_k) = w$ (say). By LSM, z_i covers $z_i \wedge z_j$ and $z_i \wedge z_k$, hence is their sup. Now $\sigma(z_i \wedge z_j) = \sigma(z_i) \wedge \sigma(z_j) = \sigma(z_i)$. Hence by Proposition 2(ii), $\sigma(z_i \wedge z_k) = \sigma(w)$; and similarly $\sigma(z_j \wedge z_k) = \sigma(w)$. We thus have $\sigma(z_i \wedge z_k) = \sigma(z_j \wedge z_k) = \sigma(w)$. But since z_k covers $z_i \wedge z_k$ and $z_j \wedge z_k$ and is their sup, we have $\sigma(z_k) = \sigma(z_i \wedge z_k) \vee \sigma(z_j \wedge z_k) = \sigma(w) \vee \sigma(w) = \sigma(w)$. Hence $\sigma(z_k) = \sigma(w)$, which $\leq \sigma(z_i)$ since $w < z_i$, and we are done.

We next show that if $\sigma(z_h)$ is not maximal, it is covered by a maximal element. In fact, let $\sigma(z_h) < \sigma(z_i)$, where $\sigma(z_i)$ is maximal; thus $\hat{z}_h < \hat{z}_i \leq z_i$. Now $\hat{z}_h \leq z_i \wedge z_h \leq z_h$, while $\sigma(\hat{z}_h) = \sigma(z_h)$; hence $\sigma(z_i \wedge z_h) = \sigma(z_h)$. Since z_i covers $z_i \wedge z_h$, by Proposition 4 we have $\dim(\sigma(z_i) : \sigma(z_h)) = \dim(\sigma(z_i) : \sigma(z_i \wedge z_h)) \leq 1$.

We can now show that if any $\sigma(z_h)$ is not maximal, say $\sigma(z_i)$ covers $\sigma(z_h)$ with $\sigma(z_i)$ maximal, then $\sigma(z_i)$ is the unique maximal

element. Indeed, as in the preceding paragraph we have $\sigma(z_i \wedge z_h) = \sigma(z_h)$. Thus for any k we have $\sigma((z_i \wedge z_h) \wedge (z_k \wedge z_h))$ (call it $\sigma(w)$) $= \sigma(z_h) \wedge \sigma(z_k \wedge z_h) = \sigma(z_k \wedge z_h)$, and since w and $z_k \wedge z_h$ each $\leq z_k$, this implies $\sigma(w) = \sigma(z_k \wedge z_h)$. Now $w \leq z_i \wedge z_k \leq z_k$, so that $\sigma(z_k \wedge z_h) = \sigma(w) \leq \sigma(z_i \wedge z_k) \leq \sigma(z_k)$, and since z_k covers $z_k \wedge z_h$, by Proposition 4 one of the $\leq$'s must be $=$. Suppose that $\sigma(w) = \sigma(z_i \wedge z_k)$. Since z_k covers $z_i \wedge z_k$ and $z_k \wedge z_h$ and is their sup, we have $\sigma(z_k) = \sigma(z_i \wedge z_k) \vee \sigma(z_k \wedge z_h) = \sigma(w) \vee \sigma(w) = \sigma(w)$. Hence in any case we have $\sigma(z_i \wedge z_k) = \sigma(z_k)$, and since $z_i \wedge z_k < z_i$, this implies that $\sigma(z_k) = \sigma(z_i \wedge z_k) \leq \sigma(z_i)$ for all k .

We have thus shown that either $\sigma(z_1), \dots, \sigma(z_m)$ are all distinct and all maximal, or else $\sigma(\mathcal{L}^*)$ has a greatest element $\sigma(z_i)$, and any $\sigma(z_h)$'s which are not equal to $\sigma(z_i)$ are covered by it and distinct. In the first case, we adjoin a new element $1' = \sigma(1)$ to $\sigma(\mathcal{L}^*)$, and define $\sigma(x) \leq 1'$ for all x ; thus $1'$ covers the $\sigma(z_i)$'s. In the second case, we simply define $\sigma(1) = \sigma(z_i)$. In either case, the extended σ is a meet homomorphism, since for all x we have $\sigma(x \wedge 1) = \sigma(x) = \sigma(x) \wedge \sigma(1)$. In the first case, we also have $\sigma(z_r \vee z_s) = \sigma(1) = 1' = \sigma(z_r) \vee \sigma(z_s)$, so that σ is an LSH. Similarly, in the second case, $\sigma(z_r \vee z_s) = \sigma(1) = \sigma(z_i)$, and since $\sigma(z_r)$ and $\sigma(z_s)$ are either equal to, or distinct and covered by, $\sigma(z_i)$, we always have $\sigma(z_i) = \sigma(z_r) \vee \sigma(z_s)$. $\square$

COROLLARY 13. *If $\mathcal{L}$ is a finite G-lattice having n atoms, there exist exactly 2^n LSH's on $\mathcal{L}$ which are all inequivalent (in the sense of Theorem 7).*

In addition, it can be shown that if a relatively complemented finite dimensional lattice $\mathcal{L}$ has *unique* relative complements then it is both modular and $\mathcal{A}$ -regular. Hence both Theorems 12 and 3 are applicable and we have a sufficient condition for the existence of lattice homomorphisms in the case of relatively complemented lattices.

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