ON THE PRODUCT OF SINGULAR SYMMETRIC MATRICES

LOUIS BRAND

The following theorem has been conjectured and used in statistical applications, but, so far as known, has not been proved.

THEOREM. *A and B are real $n \times n$ symmetric matrices with $\lambda_1, \dots, \lambda_r, 0, \dots, 0$ and $0, \dots, 0, \lambda_{r+1}, \dots, \lambda_n$ ($\lambda_1 \neq 0$) as eigenvalues respectively. If $A+B$ has $\lambda_1, \lambda_2, \dots, \lambda_n$ as eigenvalues, then $AB=0$.*

PROOF. If C is a symmetric matrix having $\lambda_1, \dots, \lambda_n$ as eigenvalues, then C can be put in the dyadic form [1]

$$(1) \quad C = \lambda_1 c_1 c_1^i + \dots + \lambda_n c_n c_n^i, \quad c^i c_j = \delta_j^i,$$

where the c_i are unit column vectors that form an orthonormal set and c^i is the row vector transpose of c_i . The n dyads $c_i c^i$ are $n \times n$ matrices of rank 1, and are precisely the n idempotents of C , whereas the products $c^i c_j$ are 1×1 matrices, the numbers δ_j^i .

Thus A and B have the dyadic form

$$A = \lambda_1 a_1 a_1^i + \dots + \lambda_r a_r a_r^i, \quad a^i a_j = \delta_j^i,$$

$$B = \lambda_{r+1} b_{r+1} b_{r+1}^i + \dots + \lambda_n b_n b_n^i, \quad b^i b_j = \delta_j^i.$$

Any matrix in dyadic form can be factored:

$$M = p_1 q^1 + \dots + p_n q^n = PQ$$

where P has $p_1, \dots, p_n$ as columns, Q has $q^1, \dots, q^n$ as rows; and $\det M = \det P \cdot \det Q$. Therefore $\det(A+B) = \lambda_1 \lambda_2 \dots \lambda_n \Delta^2$ where Δ is the determinant whose rows are the *unit* vectors $a^1, \dots, a^r, b^{r+1}, \dots, b^n$. But when the symmetric matrix $A+B$ is put in the form (1), $\det(A+B) = \lambda_1 \lambda_2 \dots \lambda_n$; and since $\lambda_i \neq 0$, this implies $\Delta^2 = 1$. By the Hadamard determinant theorem,

$$\Delta^2 \leq |a^1|^2 \dots |a^r|^2 |b^{r+1}|^2 \dots |b^n|^2 = 1$$

with equality when, and only when, the vectors are mutually orthogonal; thus $a^i b_j = 0$ ($1=i, \dots, r; j=r+1, \dots, n$) and $AB=0$.

REFERENCES

1. M. Pease, *Methods of matrix algebra*, Academic Press, New York, 1965.

UNIVERSITY OF HOUSTON

Received by the editors December 10, 1968.