($-1, 1$) RINGS

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The center of a nonassociative ring is the set of all elements which commute and associate with all other elements of the ring, i.e. $\{x \mid [x, a] = (x, a, b) = (a, x, b) = (x, a, b) = 0 \text{ for all elements } a, b\}$. Hypotheses on the ring imply conditions on the center, e.g. the center of any simple nonassociative ring is either 0 or a field. We examine the reverse problem. We place hypotheses on the center and see how they are reflected in properties of the ring.

A $(-1, 1)$ ring is a nonassociative ring in which the following identities are assumed to hold:

- (1) $(a, b, c) + (a, c, b) = 0$,
- (2) $(a, b, c) + (b, c, a) + (c, a, b) = 0$

for all elements a, b, c . We study $(-1, 1)$ rings with an idempotent $\neq 0, 1$ with the further property that their centers contain no trivial ideals. (An ideal I is trivial if $I \neq 0$ but $I^2 = 0$.) Except for the cases of characteristic 2 and 3, we will be able to show that such an algebra has a Peirce decomposition $A = A_{11} + A_{10} + A_{01} + A_{00}$ where the multiplication table for the summands is the same as if A were associative. We will also show that $A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10}$ is an ideal of A contained in the nucleus of A . (The nucleus of a ring A is $\{x \mid (a, b, x) = (a, x, b) = (a, b, x) = 0 \text{ for all } a, b \in A\}$.) This latter statement was proved by Sterling [5] using the stronger hypothesis that the ring was free of trivial ideals entirely.

We also solve a different type of question on $(-1, 1)$ rings. A nil power associative algebra is an algebra in which for every element x , there exists an integer $n(x)$ such that $x^{n(x)} = 0$. An algebra is nilpotent if there exists n such that $A^n = 0$.

There has been much work on nil and nilpotency conditions on finite-dimensional algebras. It is known that nil finite-dimensional Jordan algebras are nilpotent and that nil finite-dimensional alternative algebras are nilpotent. It is unknown if nil finite-dimensional right alternative algebras are nilpotent. We will show that nil finite-dimensional $(-1, 1)$ algebras such that $2x = 0$ implies $x = 0$ are nilpotent. In Albert's paper [1] he showed that every nil finite-dimensional (γ, δ) algebra was nilpotent except for the cases

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$(\gamma + \delta)(\gamma + \delta - 1) = 0$. Thus, proving the result for $(-1, 1)$ rings partially fills in the gap.

Preliminaries. A right alternative ring is a nonassociative ring satisfying (1). When we say a ring has characteristic not n , n an integer, we mean that $nx = 0$ implies $x = 0$ for all elements x of the ring. The following identities hold in any right alternative ring of characteristic not 2:

- (3) $(x, y^r, y^s) = 0$ for all positive integers r, s ,
- (4) $x(yz + zy) = (xy)z + (xz)y$,
- (5) $(x, yz, y) = (x, z, y)y$,
- (6) $(x, yz, w) + (x, wz, y) = (x, z, w)y + (x, z, y)w$,
- (7) $[(xy)z]y = x[(yz)y]$,
- (8) $(x, y^2, z) = (x, y, yz + zy)$,
- (9) $(ab, c, d) - (a, bc, d) + (a, b, cd) = a(b, c, d) + (a, b, c)d$.

See [3] for the proofs of (5), (6), (7), (8), and (9). (4) is direct from (1) and (3) is an easy induction using (6).

LEMMA 1 (ALBERT). *Let A be a right alternative ring with characteristic not 2 and with an idempotent e . Then A may be written as $A = A_1 + A_{1/2} + A_0$ (additive direct sum) where $x \in A_1 \Leftrightarrow ex = xe = x$; $x \in A_0 \Leftrightarrow ex = xe = 0$; $x \in A_{1/2} \Leftrightarrow ex + xe = x$. Furthermore, $[x, e] = 0 \Leftrightarrow x \in A_1 + A_0$; $A_1A_0 = A_0A_1 = 0$.*

PROOF. If $x \in A$, let $x_1 = (ex)e$, let $x_{1/2} = ex + xe - 2x_1$, let $x_0 = x - x_1 - x_{1/2}$; $ex_1 = x_1$ by (7); $x_1e = x_1$ by (3). Thus $ex_1 = x_1e = x_1$. From $ex_1 = x_1e = x_1$ and (4), it is easy to see $ex_{1/2} + x_{1/2}e = x_{1/2}$ and $ex_0 = x_0e = 0$. Let $A_1 = \{(ex)e \mid x \in A\}$. Let $A_{1/2} = \{ex + xe - 2(ex)e \mid x \in A\}$. Let $A_0 = \{x - (ex + xe) + (ex)e \mid x \in A\}$. It is clear that $A_1 + A_{1/2} + A_0 = A$. The proof that this sum is additively direct follows from the multiplicative properties of the subgroups and the idempotent e . To prove that $[x, e] = 0 \Leftrightarrow x \in A_1 + A_0$, we first note that

$$(10) \quad x = 2xe \text{ implies } x = 0.$$

Iterating the relation $x = 2xe$ gives $x = 2xe = 4(xe)e = 4xe$, which quickly leads to $x = 0$. It is then immediate that $[x, e] = 0$ leads to $0 = [x_{1/2}, e]$. So $x_{1/2} = ex_{1/2} + x_{1/2}e = 2x_{1/2}e$. Thus $x_{1/2} = 0$. The proof that $A_1A_0 = A_0A_1 = 0$ follows from (11) below.

$$(11) \quad (A, e, A_1 + A_0) = 0,$$

$$(12) \quad (eA_{1/2})e = 0.$$

(11) is from (8) and the multiplicative properties of e on A_1 and A_0 . (12) comes since

$$\begin{aligned} x_{1/2}e &= (ex_{1/2} + x_{1/2}e)e = (ex_{1/2})e + (x_{1/2}e)e = (ex_{1/2})e + x_{1/2}(ee) \quad \text{by (3)} \\ &= (ex_{1/2})e + x_{1/2}e. \end{aligned}$$

(-1, 1) rings with an idempotent. In this section all rings are $(-1, 1)$ rings with an idempotent $e \neq 0, 1$ and with characteristic not 2 or 3. By the symbol $[x, y]$ we will mean $[x, y] = xy - yx$. We shall use these identities from Maneri [4].

$$(13) [a, (b, c, d)] - [b, (c, d, a)] + [c, (d, a, b)] - [d, (a, b, c)] = 0,$$

$$(14) [(e, e, A), A] = 0,$$

$$(15) (e, e, x)^2 = 0.$$

(13) is statement G in [4]; (14) is statement (14) in [4]; (15) is statement (12) in [4].

LEMMA 2. $(e, e, A)A_{1/2} = 0$.

PROOF.

$$\begin{aligned} [(e, e, a)e]a_{1/2} &= (e, e, ea)a_{1/2} \text{ by (5)} \\ &= a_{1/2}(e, e, ea) \text{ by (14)} \\ &= a_{1/2}[(e, e, a)e] \text{ by (5)} \\ &= [a_{1/2}(e, e, a)]e \text{ by (11)} \end{aligned}$$

[since $[(e, e, A), e] = 0$ by (14) implies $(e, e, A) \in A_1 + A_0$]

$$= ((e, e, a)a_{1/2})e \text{ by (14).}$$

Now $(e, e, a)a_{1/2} = (e, e, a)(ea_{1/2} + a_{1/2}e)$ by Lemma 1

$$\begin{aligned} &= [(e, e, a)e]a_{1/2} + [(e, e, a)a_{1/2}]e \text{ by (4)} \\ &= 2[(e, e, a)a_{1/2}]e \text{ by the first sentence of this proof.} \end{aligned}$$

Thus $(e, e, a)a_{1/2} = 2[(e, e, a)a_{1/2}]e = 0$ by (10).

LEMMA 3. $(e, e, x)a = (e, e, (a_1 + a_0)x)$ where $a = a_1 + a_{1/2} + a_0$.

PROOF. $(e, e, x)a = (e, e, x)(a_1 + a_0)$ by Lemma 2. Also

$$(e, e, x)(a_1 + a_0) + (e, a_1 + a_0, x)e = (e, e, (a_1 + a_0)x) + (e, a_1 + a_0, ex)$$

by (6). To prove Lemma 3 it suffices to show $(e, a_1 + a_0, x)e = (e, a_1 + a_0, ex)$. $(e, a_1 + a_0, x)e = [-(a_1 + a_0, x, e) - (x, e, a_1 + a_0)]e$ by (2) $= -(a_1 + a_0, x, e)e$ by (11) $= -(a_1 + a_0, ex, e)$ by (5) $= (ex, e, a_1 + a_0) + (e, a_1 + a_0, ex)$ by (2) $= (e, a_1 + a_0, ex)$ by (11).

LEMMA 4. $(e, A, A) \subset (e, e, A)$.

PROOF. $x(u, x, u) = x[(ux)u] - x[u(xu)] = [(xu)x]u - x[u(xu)]$ by (7) $= (xu, x, u) + (x, u, xu) = -(u, xu, x)$ by (2) $= -(u, u, x)x$ by (5) $= (u, x, u)x$ by (1). Thus

$$(16) [x, (u, x, u)] = 0$$

and by linearizing we get

(17) $[x, (b, a, x)] = -[x, (a, b, x)]$.
 $[a, (x, x, y)] - [x, (x, y, a)] + [x, (y, a, x)] - [y, (a, x, x)] = 0$ by (13).
 Thus $[a, (x, x, y)] = -3[x, (y, a, x)]$ by (2), (17) and (3). $0 = [a, (e, e, y)]$ by (14) $= -3[e, (y, a, e)]$. Thus $[e, (e, y, a)] = 0$ by (2) and (1). Thus $e(e, y, a) = (e, y, a)e$. $(e^2, x, y) - (e, ex, y) + (e, e, xy) = e(e, x, y) + (e, e, x)y$ by (9). $(e, ex, y) + (e, yx, e) = (e, x, y)e + (e, x, e)y$ by (6). Adding these expressions, remembering $e^2 = e$, we get $(e, x, y) + (e, e, xy - yx) = e(e, x, y) + (e, x, y)e$. Since $e(e, x, y) = (e, x, y)e$, we get $(e, x, y) + (e, e, xy - yx) = 2(e, x, y)e$. Multiplying this equation by e on the right gives $(e, x, y)e = (e, e, [x, y])e = (e, e, e[x, y])$ by (5). Thus $(e, x, y) = 2(e, x, y)e - (e, e, [x, y]) = 2(e, e, e[x, y]) - (e, e, e[x, y]) + [x, y]e$ by (8) $= (e, e, e[x, y]) - [x, y]e = (e, e, [e, [x, y]])$.

COROLLARY.

$$(18) [A, (e, A, A)] = 0.$$

The proof is from Lemma 4 and (14).

LEMMA 5. $(A, A, A) \subset A_1 + A_0$.

PROOF. By Lemma 1, $a \in A_1 + A_0 \Leftrightarrow [e, a] = 0$. We will show $[e, (a, b, c)] = 0$ for all a, b, c . Because $[e, (a, b, c)] - [a, (b, c, e)] + [b, (c, e, a)] - [c, (e, a, b)] = 0$ by (13), by (18) it suffices to show (19) $[a, (b, c, e)] = 0$ for all a, b, c .
 $[a, (b, c, e) + (c, e, b) + (e, b, c)] = 0$ by (2). Thus $[a, (b, c, e)] = [a, (c, b, e)]$ by (18). Since $[a, (b + c, b + c, e)] = [a, (b, b, e)] + [a, (b, c, e)] + [a, (c, b, e)] + [a, (c, c, e)]$, we have $2[a, (b, c, e)] = [a, (b + c, b + c, e)] - [a, (b, b, e)] - [a, (c, c, e)]$. Hence it suffices to show $[a, (x, x, e)] = 0$ for all a, x . $[a, (x, x, e)] - [x, (x, e, a)] + [x, (e, a, x)] - [e, (a, x, x)] = 0$ by (13). So $[a, (x, x, e)] = [x, (x, e, a)]$ by (18). $[x, (x, e, a)] = -[x, (e, a, x)] - [x, (a, x, e)]$ by (2) $= -[x, (a, x, e)]$ by (18). By linearization of (16) we have $[x, (a, x, e)] = -[x, (e, x, a)] = 0$ by (18). This finishes the proof of Lemma 5.

COROLLARY TO LEMMA 5. $[A, (A, A, e)] = 0$.

This is statement (19).

LEMMA 6. (e, e, A) is a trivial ideal in the center of A .

PROOF. By (14) $[A, (e, e, A)] = 0$. By Lemma 3 it is an ideal, by (15) and (14) it is a trivial ideal. It suffices to show (e, e, A) is in the nucleus of A . By (1) and (2), it suffices to show $(a, (e, e, x), b) = 0$ for all a, x, b .

$$\begin{aligned}
(a, (e, e, x), b) &= (a_1 + a_0, (e, e, x), b_1 + b_0) \quad \text{by Lemmas 2 and 3} \\
&= - (a_1 + a_0, b_1 + b_0, (e, e, x)) \\
&= - (e, e, (a_1 + a_0, b_1 + b_0, x)) \quad \text{by Lemma 3}
\end{aligned}$$

(and the easy result that $(A_1 + A_0)(A_1 + A_0) \subset A_1 + A_0 = 0$ by Lemma 5.

THEOREM 1. *If A is a $(-1, 1)$ ring of characteristic not 2 or 3 with an idempotent e and no trivial ideals in its center, then A has a decomposition $A = A_{11} + A_{10} + A_{01} + A_{00}$ where $ea_{ij} = \delta_{1i}a_{ij}$ and $a_{ij}e = a_{ij}\delta_{j1}$. Moreover, a multiplication table for the summands is the same as if A were associative.*

PROOF. Assume A has no trivial ideals in the center. From Lemma 4 and Lemma 6 we have $(e, A, A) = 0$, $A = A_1 + A_{1/2} + A_0$. Let $A_{11} = A_1$, $A_{00} = A_0$, $A_{10} = eA_{1/2}$, $A_{01} = A_{1/2}e$. It is clear that $a_{ij} \in A_{ij}$ if and only if $ea_{ij} = \delta_{1i}a_{ij}$ and $a_{ij}e = \delta_{j1}a_{ij}$ where δ_{ij} is the Kronecker delta; $\delta_{ii} = 1$, $\delta_{ij} = 0$, $i \neq j$. Furthermore $e(ex_{1/2}) + (ex_{1/2})e = (ee)x_{1/2} + 0 = ex_{1/2}$. Thus $A_{10} \subset A_{1/2}$. So A_{10} and $A_{01} \subset A_{1/2}$. This, and $x_{1/2} = ex_{1/2} + x_{1/2}e$, implies $A_{10} + A_{01} = A_{1/2}$. The sum is additively direct because $ex_{1/2} + y_{1/2}e = 0 \Rightarrow 0 = (ex_{1/2} + y_{1/2}e)e = ex_{1/2} \cdot e + y_{1/2}e = y_{1/2}e$ by (12). Thus $y_{1/2}e = 0$ and then also $ex_{1/2} = 0$. Now for the table.

Suppose $(x_{ij}, y_{kl}, e) \neq 0$. By (11), $k \neq l$. Since $0 = (x_{ij}, y_{kl}, e) + (y_{kl}, e, x_{ij}) + (e, x_{ij}, y_{kl})$ by (2) $= (x_{ij}, y_{kl}, e) + (y_{kl}, e, x_{ij})$, we have $i \neq j$. $(x_{ij}, y_{kl}, e) = -(x_{ij}, e, y_{kl}) = x_{ij}y_{kl}(-\delta_{lj} + \delta_{lk})$. Thus $j \neq k$. We have shown the only time an associator of the form $(x_{ij}, y_{jk}, e) \neq 0$ is when it is of the type (x_{10}, y_{10}, e) or (x_{01}, y_{01}, e) . This gives us the table except for the products $A_{10}A_{10}$ and $A_{01}A_{01}$. $a_{10}, b_{10} = -(a_{10}, e, b_{10})$ so by (19) $[(a_{10}, e, b_{10}), A] = 0$. Since $[e, a_{10}b_{10}] = 0$, $a_{10}b_{10} \in A_{11} + A_{00}$; $(a_{10}, e, b_{10})e = (a_{10}, e, eb_{10})$ by (5) $= (a_{10}, e, b_{10})$. The preceding sentence shows $a_{10}b_{10} \in A_{11}$. $(a_{10}b_{10})x_{01} = 0$, $(a_{10}b_{10})x_{10} = x_{10}(a_{10}b_{10}) \subset A_{10}A_{11} = 0$. Thus:

$$(20) \quad (A_{10}A_{10})A_{1/2} = A_{1/2}(A_{10}A_{10}) = 0.$$

Similarly $(A_{01}A_{01})A_{1/2} = A_{1/2}(A_{01}A_{01}) = 0$. Because

$$\begin{aligned}
a_{11}(b_{10}c_{10}) &= (b_{10}c_{10})a_{11} = b_{10}(c_{10}a_{11}) - (b_{10}, a_{11}, c_{10}) \\
&= b_{10}(a_{11}c_{10}) \subset (A_{10})^2,
\end{aligned}$$

$A_{10}A_{10}$ is an ideal of A . The previous sentence also shows

$$(a_{11}, b_{11}, x_{10}y_{10}) = x_{10}(a_{11}, b_{11}, y_{10}) = -x_{10}(a_{11}, y_{10}, b_{11}) = 0;$$

thus $(A_{10})^2$ is an ideal in the center of A . Because $(x_{10}y_{10})(z_{10}w_{10})$

$=x_{10}[y_{10}(z_{10}w_{10})]-(x_{10}, z_{10}w_{10}, y_{10})=0$ by (20), we have $(A_{10})^2$ is a trivial ideal. Therefore $A_{10}A_{10}$ is a trivial ideal in the center of A ; therefore $(A_{10})^2=0$. Similarly $(A_{01})^2$ is also a trivial ideal in the center of A and therefore $(A_{01})^2=0$.

We can now prove the major lemma in Sterling's paper with weaker hypotheses.

THEOREM 2. *Let A be a $(-1, 1)$ ring with characteristic not 2 or 3, with an idempotent $e \neq 0, 1$, and with no trivial ideals in the center. Then $A = A_{11} + A_{10} + A_{01} + A_{00}$ and $A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10}$ is an ideal in the nucleus.*

PROOF. It will suffice to show $A_{1/2}$ is in the nucleus. Assume $(A_{ij}, A_{jk}, A_{kl}) \neq 0$. By (1) $j=k$ and $l=j$, so it is of the form $(A_{ij}A_{jj}, A_{jj})$. $(A_{ij}, A_{jj}, A_{jj}) \subset (A_{jj}, A_{jj}, A_{ij}) + (A_{jj}, A_{ij}, A_{jj})$ by (2); so $j=i$. We have shown $(A_{ij}, A_{kl}, A_{mn}) \neq 0 \Rightarrow i=j=k=l=m=n$. Thus any associator involving A_{10} or A_{01} is zero. Thus $A_{1/2}$ is in the nucleus.

Nil finite-dimensional $(-1, 1)$ algebras. We now turn to a different problem on $(-1, 1)$ rings. We will prove that nil finite-dimensional $(-1, 1)$ algebras are nilpotent.

For the proof, we will need a theorem by Jacobson [2]. Let A be an associative algebra and $S \subset A$ such that for any pair $s, s' \in S$, there exists a $\delta(s, s')$ in the field such that $ss' + \delta(s, s')s's \in S$. S is said to be weakly closed.

THEOREM 3 (JACOBSON [2]). *If S is a weakly closed subset of a finite-dimensional associative algebra and every element in S is nilpotent, then the algebra generated by S in A is nilpotent.*

If A is a nonassociative ring, then the set of all endomorphisms on the additive structure of A is a ring. For any $x \in A$, we define the endomorphism $L_x(R_x)$ by $aL_x = xa(aR_x = ax)$ for all $a \in A$.

LEMMA 7. *If A is a nil finite-dimensional $(-1, 1)$ algebra such that $2x=0$ implies $x=0$, then $A^2 \neq A$.*

PROOF. The Map $x \rightarrow R_x$ is a Jordan homomorphism. Thus the set $S = \{R_x | x \in A\}$ is a weakly closed subset of the finite-dimensional algebra of all linear transformations of A ($R_xR_y + R_yR_x = R_{xy+yx}$ by (4)). By Theorem 3, the subring generated by S is nilpotent. Thus $((AA)A)A \cdots A = 0$. Thus $A^2 \neq A$.

LEMMA 8. *If A is a $(-1, 1)$ ring such that $2x=0$ implies $x=0$, then $(x^r, x^s, x^t a) = x^t(x^r, x^s, a) = (x^r, x^s, a)x^t$.*

PROOF. First we will show

$$(21) \quad L_x^r L_x^s = L_x^s L_x^r.$$

$$(a, x^r, x^s) + (x^r, x^s, a) + (x^s, a, x^r) = 0 \text{ by (2). } (a, x^r, x^s) = 0 \text{ by (3).}$$

Thus

$$(22) \quad (x^r, x^s, a) = (x^s, x^r, a).$$

Expanding (22) gives $x^{r+s}a - x^r(x^s a) = x^s(x^r a) - x^s(x^r a)$. This completes the proof of (21). Now

$$\begin{aligned} (x^r, x^s, x^t a) &= a(L_x^t(L_x^{r+s} - L_x^s L_x^r)) = a((L_x^{r+s} - L_x^s L_x^r)L_x^t) \\ &= x^t(x^r, x^s, a). \end{aligned}$$

We have shown:

$$(23) \quad (x^r, x^s, x^t a) = x^t(x^r, x^s, a).$$

To finish the proof of Lemma 8, it suffices to show $[x^t, (x^r, x^s, a)] = 0$. $[x^t, (x^r, x^s, a)] - [x^r, (x^s, a, x^t)] + [x^s, (a, x^t, x^r)] - [a, (x^t, x^r, x^s)] = 0$ by (13). Thus $[x^t, (x^r, x^s, a)] = -[x^r, (x^s, x^t, a)]$ by (1) and (3). Thus one cyclic permutation of x^t, x^r, x^s changes the sign. Applying 3 permutations we have $[x^t, (x^r, x^s, a)] = -[x^t, (x^r, x^s, a)]$. Therefore $2[x^t, (x^r, x^s, a)] = 0$. So $[x^t, (x^r, x^s, a)] = 0$. This finishes the proof of Lemma 8.

LEMMA 9. If $x^N = 0$ then $\langle \{Lx^i, Rx^i \mid i = 1, 2, 3 \dots\} \rangle^{3N} = 0$. That is, the left and right multiplications by powers of a single element generate a nilpotent ring.

PROOF. Because $L_{ab} - L_b L_a + L_a R_b - R_b L_a = 0$ by (1), it will suffice to show any product $Lx^{\alpha(1)} Lx^{\alpha(2)} \dots Lx^{\alpha(s)} Rx^{\alpha(s+1)} = 0$, where $\alpha(1) + \alpha(2) + \dots + \alpha(s+1) \geq 3N$. Assume to contrary. We may then choose a nonzero product of the above type with the fewest terms. That is, any product $Lx^{\beta(1)} Lx^{\beta(2)} \dots Lx^{\beta(t)} Rx^{\beta(t+1)}$ where $\beta(1) + \beta(2) + \dots + \beta(t+1) \geq 3N$ and $t < s$ is zero. It is clear that $s \geq 3$. However

$$\begin{aligned} Lx^{\alpha(1)} Lx^{\alpha(2)} \dots Rx^{\alpha(s+1)} \\ &= (Lx^{\alpha(1)+\alpha(2)} + (Lx^{\alpha(1)} Lx^{\alpha(2)} - Lx^{\alpha(1)+\alpha(2)})) Lx^{\alpha(3)} \dots Rx^{\alpha(s+1)} \\ &= Lx^{\alpha(1)+\alpha(2)} Lx^{\alpha(3)} \dots Rx^{\alpha(s+1)} \\ &\quad + (Lx^{\alpha(1)} Lx^{\alpha(2)} - Lx^{\alpha(1)+\alpha(2)}) Rx^{\alpha(3)+\dots+\alpha(s+1)} \\ &= 0 \end{aligned}$$

because $Lx^{\alpha(1)+\alpha(2)} Lx^{\alpha(3)} \dots Rx^{\alpha(s+1)}$ is a shorter product, and the other comes from Lemma 8, (3), and $x^N = 0$.

THEOREM 4. If A is a finite-dimensional nil $(-1, 1)$ algebra such that $2x = 0$ implies $x = 0$, then A is nilpotent.

PROOF. The proof proceeds by induction on the dimension of A over the field F . If the dimension of A is 1, it is clear. Assume result is true when dimension of $A < n$. By Lemma 7, $A^2 \neq A$. So we may write $A = B + Fw$ where $A^2 \subset B \subsetneq A$, and B is a nilpotent ideal. Furthermore, if B^k means the additive subgroup generated by all products of k elements taken from B , then B^k is an ideal of A . We will not prove the above statement. Suppose $B^k = 0$, $w^N = 0$. Any product of 2^{3Nk} elements of A can be written as a sum of products of the form $T_1 \cdots T_{3N} T_{3N+1} \cdots T_{6N} T_{6N+1} \cdots T_{9N} T_{9N+1} \cdots T_{3N(K-1)+1} \cdots T_{3NK}$ where the T_i 's are either left or right multiplications by elements of B or by w . If k of the T_i 's are multiplications by elements of B , then the image is in $B^k = 0$. If not, then one of the N blocks $T_{3r+1} \cdots T_{3(r+1)}$ in the product is left or right multiplications by L_w or R_w and this is zero since we have previously shown that $\langle \{L_{x^i}, R_{x^i} \mid i = 1, 2, \dots\} \rangle^{3N} = 0$ if $x^N = 0$ in Lemma 9. This completes the proof of Theorem 4.

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