LOCALLY FINITE AND SOLVABLE SUBGROUPS OF SFIELDS

R. J. FAUDREE¹

In this paper we will determine all locally finite subgroups of a sfield and give a criterion for determining when the group algebra of an ascending solvable torsion free group has a quotient sfield.

NOTATION AND DEFINITIONS. A group G has property E if it can be embedded in a sfield D and property EE if every automorphism of the group G can be extended to be an automorphism of D . For a more complete discussion see [5].

The result on locally finite groups depends mainly on a result of Amitsur (see [1]). A complete discussion of the following notation can be found in his paper.

π will denote the set of all primes and π_1 the set of all odd primes p such that 2 has odd order mod p . Let m and r be relatively prime integers. Put $s = (r-1, m)$, $t = m/s$ and $n =$ minimal integer satisfying $r^n \equiv 1 \pmod{m}$. Denote by $G_{m,r}$ a group generated by two elements A and B satisfying $A^m = 1$, $B^n = A^t$ and $BAB^{-1} = A^r$. Denote by $G_{m,r}^\infty$ a group G which has a countable ascending tower of subgroups $\{H_i: 0 \leq i < \infty\}$ such that $G = \bigcup_{i=1}^\infty H_i$ and each H_i is isomorphic to G_{m_i, r_i} . T^* , O^* , I^* will denote the binary tetrahedral, octahedral and icosahedral groups.

Let p be a fixed prime dividing m .

$\alpha = \alpha_p$ is the highest power of p dividing m .

η_p is the minimal integer satisfying $r^{\eta_p} \equiv 1 \pmod{mp^{-\alpha}}$.

μ_p is the minimal integer satisfying $r^{\mu_p} \equiv p^{\mu'} \pmod{mp^{-\alpha}}$ for some integer μ' .

δ_p is the minimal integer such that $p^{\delta_p} \equiv 1 \pmod{mp^{-\alpha}}$.

$\delta'_p = \mu_p \delta_p / \eta_p$.

CONDITION C. Integers m and r satisfy Condition C if either

(I) $(n, t) = (s, t) = 1$, or

(II) $n = 2n'$, $m = 2^\alpha m'$, $s = 2s'$ where $\alpha \geq 2$, m' , s' and n' are odd integers; $(n, t) = (s, t) = 2$ and $r \equiv -1 \pmod{2^\alpha}$. And either

(III) $n = s = 2$ and $r \equiv -1 \pmod{m}$, or

(IV) for every q/n there exists a prime $p \mid m$ such that $q \nmid \eta_p$ and that either

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- (1) $p \neq 2$ and $(q, (p^{s'} - 1)/s) = 1$, or
- (2) $p = q = 2$, (II) holds and $m/4 \equiv \delta'_2 \equiv 1 \pmod{2}$.

Amitsur proved the following (see [1]):

THEOREM 1. *A finite group has property E if and only if G is isomorphic to one of the following groups:*

- (i) cyclic group,
- (ii) $G_{m,r}$ where m and r satisfy Condition C,
- (iii) $T^* \times G_{m,r}$ where $G_{m,r}$ is either cyclic of order m or of type (2), $(G, |G_{m,r}|) = 1$ and $p \mid m$ implies p in π_1 ,
- (iv) O^* and I^* .

We will prove the following generalization of Amitsur's Theorem.

THEOREM 2. *Let G be a locally finite group. The following are equivalent:*

- (1) G has property E,
- (2) G has property EE,
- (3) G is isomorphic to
 - (i) a subgroup of $\prod_{p \in \pi} Z(p^\infty)$,
 - (ii) $G_{m,r}^\infty$ where m_i and r_i satisfy Condition C,
 - (iii) $T^* \times H$ where either (a) H is a subgroup of $\prod_{p \in \pi_1} Z(p^\infty)$,
- (b) $H = G_{m,r}^\infty$ where $(|G_{m_i, r_i}|, 6) = 1$, $p \mid m_i$ implies p is in π_1 and m_i and r_i satisfy Condition C,
- (iv) O^* and I^* .

Theorem 2 for countable locally finite groups was proved in [6]. Thus to prove Theorem 2 it is sufficient to show that any locally finite subgroup of a sfield is countable.

Let G be a locally finite group which can be embedded in a division ring. If S is a subset of G then $\langle S \rangle$ will denote the subgroup of G generated by S .

LEMMA 1. *If G has a subgroup isomorphic to O^* or I^* then G is isomorphic to O^* or I^* .*

PROOF. O^* and I^* are the only groups satisfying Theorem 1 which are not solvable of length 3 or less.

In the remaining lemmas it will be assumed that G has no subgroups isomorphic to O^* or I^* .

LEMMA 2. *If G has T^* as a subgroup then there is a subgroup H of G such that $G = T^* \times H$ and every finite subgroup of H satisfies (I) of Condition C.*

PROOF. Let $R = C_G(T^*)$. Let $H = \{x \in C_G(T^*) \mid |x| \text{ is odd}\}$. If

$x, y \in H$ and $g \in G$, then $\langle x, y, g, T^* \rangle = T^* \times S$ where S is a subgroup of odd order satisfying (I) of Condition C by Theorem 1. $|Z(T^*)| = 2$, thus $x, y \in S$. Therefore H is a subgroup. Also $g = t \cdot s$ with $t \in T^*$, $s \in S \subseteq H$. Hence $G = T^* \times H$.

A *factor* of a group G is a quotient group of a subgroup of G . The *rank* of an abelian group G will be k if it has an elementary abelian factor of order p^k for some prime p but no elementary abelian factor of order p^{k+1} for some prime p . The *derived factors* of a group G are the quotient groups G_i/G_{i+1} where G_i is the i th derived group.

LEMMA 4. *If G is a locally finite group embeddable in a sfield then $G \cong K \times H$ where K is a finite group and H is solvable of length ≤ 2 and each derived factor of K is of rank ≤ 2 .*

PROOF. By Lemmas 1, 2 and Theorem 1, $G = K \times H$ with $K \cong O^*$, I^* or T^* and every finite subgroup of H is metacyclic. Since a metacyclic group is solvable of length ≤ 2 and each derived factor is of rank ≤ 2 , the same is true of H .

LEMMA 5. *Let G be a solvable group of derived length n . If for some k each of the derived factors has rank at most k , then G is countable.*

PROOF. If A is an uncountable abelian group then it does not have finite rank. Thus G_i/G_{i+1} is countable for each i and hence G is countable.

Theorem 2 is a consequence of Lemmas 4 and 5.

It follows that in a sfield there is a maximal locally finite subgroup and it is countable.

If G is a finite subgroup of a sfield D and generates D then the automorphism group of D is determined by the automorphism group of G modulo the inner automorphism group of D (see [7]). It would be interesting to know if this is also true for a locally finite group.

Before proving the next theorem we will give some more definitions and notation.

A group G is *ascending solvable* if G has an ascending normal series such that each factor is abelian [9, p. 163]. If K is a sfield and G a group, then $K[G]$ will denote the *group algebra* of G over K . If θ is an automorphism of K and x an indeterminate over K , $K[x, \theta]$ will denote the *Ore polynomial ring* in x over K determined by θ (see [11]). A ring R is *regular* if it has no divisors of zero and if for elements a and b in R there are nonzero elements a_1, b_1, a_2 and b_2 in R such that $aa_1 = bb_1$ and $a_2a = b_2b$. F is a *quotient sfield* of R if R is a subring of the sfield F and for any f in F , $f = r_1r_2^{-1} = r_3^{-1}r_4$ for r_i in R ($1 \leq i \leq 4$).

The above are connected by the following proposition proved by Asano [2].

PROPOSITION 1. *A ring R has a quotient sfield if and only if R is a regular ring. The quotient sfield is unique up to isomorphism. Any automorphism θ of R can be extended uniquely to F .*

The following result was proved by Ore [11]:

PROPOSITION 2. *$K[x; \theta]$ is a regular ring.*

The quotient sfield of $K[x; \theta]$ will be denoted by $K(x; \theta)$. If the group algebra $K[G]$ is a regular ring, its quotient sfield will be denoted by $K(G)$.

The following is a special case of a theorem of Bovdi [3]:

PROPOSITION 3. *If G is an ascending solvable group and K a sfield, then the group algebra $K[G]$ has a quotient sfield if and only if $K[G]$ has no divisors of zero.*

We will give a criterion to determine when the group algebra has no divisors of zero. The following definition and proposition is due to Ore [10].

Let R be a regular ring and $A = (a_{ij})$ be an $n \times n$ matrix with coefficients in R . We now give a determinant function $|A|$ for A .

If $n=1$, define $|A| = a_{11}$. If $n=2$, define $|A| = a_{11}a_{22} - a_{21}a_{12}$ where A_{22} and A_{12} have the property that $a_{12}a_{22} = a_{22}a_{12}$. For n define $|A| = a_{11}A_1^{(1)} + a_{21}A_1^{(2)} + \cdots + a_{n1}A_1^{(n)}$ where the $A_1^{(j)}$ are a set of solutions to the homogeneous equations

$$\begin{aligned} a_{12}A_1^{(1)} + a_{22}A_1^{(2)} + \cdots + a_{n2}A_1^{(n)} &= 0 \\ \vdots \\ a_{1n}A_1^{(1)} + a_{2n}A_1^{(2)} + \cdots + a_{nn}A_1^{(n)} &= 0. \end{aligned}$$

The function $|A|$ has the following important property:

PROPOSITION 4. *The linear system*

$$\sum_{i=1}^n x_i a_{ij} = 0 \quad (1 \leq j \leq n) \quad a_{ij} \in R$$

has a nontrivial solution if and only if $|a_{ij}| = 0$.

THEOREM 3. *Let G be an ascending solvable group with an ascending normal series $\{H_i\}_{i=0}^\alpha$ with $H_{i+1}/H_i = \langle x_{i+1}, H_i \rangle$. Let n_{i+1} be the order of $x_{i+1} \bmod H_i$ and let θ_{i+1} be the automorphism of $K[H_i]$ induced by x_{i+1} . $K[G]$ has a quotient sfield if and only if for each i that n_{i+1} is*

finite and for all n_{i+1} tuples $(d_0, \dots, d_{n_{i+1}})$ of elements of $K[H_i]$, $|a_{kj}| \neq 0$ where

$$\begin{aligned} a_{kj} &= \theta_{i+1}^j(d_{k-j}) \quad \text{if } k \geq j, \\ &= \theta_{i+1}^j(d_{n_{i+1}+k-j})x_{i+1}^{n_{i+1}} \quad \text{if } k < j. \end{aligned}$$

PROOF. By Proposition 3 it is sufficient to show that $K[G]$ has no divisors of zero. Assume that $K[H_i]$ has no divisors of zero for all $i < j \leq \alpha$. If j is a limit ordinal then $K[H_j]$ has no divisors of zero. If j is not a limit ordinal then there are two possibilities, either x_j has infinite order mod H_{j-1} or finite order mod H_{j-1} . In the first case it is obvious that $K[H_j]$ has no divisors of zero.

Therefore we are left with considering the case of G and H groups with H normal in G , $G/H = \langle x \cdot H \rangle$ where x has order n mod H and $K[H]$ has no divisors of zero. Let θ be the automorphism of $K[H]$ induced by x . Every element of $K[G]$ can be written uniquely in the form $\sum_{i=0}^{n-1} d_i x^i$ for d_i in $K[H]$. Let $w = \sum_{i=0}^{n-1} d_i x^i$ be a fixed element of $K[G]$. If w is a divisor of zero, then there is a $y = \sum_{j=0}^{n-1} y_j x^j$ with y_j in $K[H]$ such that $yw = 0$. Thus

$$\sum_{j=0}^{n-1} y_j (x^j w) = \sum_{j=0}^{n-1} y_j \left(\sum_{i=0}^n \theta^j(d_i) x^{i+j} \right) = 0.$$

This breaks down into the following system of n -equations in the unknowns y_i .

$$\begin{array}{ccccccc} y_0 d_0 + y_1 \theta(d_{n-1}) x^n + \dots + y_{n-1} \theta^{n-1}(d_1) x^n & = & 0 \\ y_0 d_1 + y_1 \theta(d_0) + & & + y_{n-1} \theta^{n-1}(d_2) x^n & = & 0 \\ \vdots & & & & \vdots \\ v_0 d_{n-1} + y_1 \theta(d_{n-2}) + \dots + y_{n-1} \theta^{n-1}(d_0) & = & 0. \end{array}$$

Application of Proposition 4 completes the proof.

The determinant given in Theorem 3 is very complicated. For $n=2$ or 3 we will give a simpler form of the same expression. This will be done in the next two lemmas.

LEMMA 6. Let K be a sfield $0 \neq b \in K$ and θ an automorphism of K such that $\theta(b) = b$ and θ^n is the same automorphism of K as the automorphism induced by b in K . Consider the ring $K[x; \theta]$. The following are equivalent.

- (1) $x^n - b$ is irreducible.
- (2) $(x^n - b)$ is a prime ideal.

(3) $K[x; \theta]/(x^n - b)$ is a sfield.

Conditions (1), (2) and (3) imply that $(x^n - b)$ is maximal but the converse is not true.

PROOF. $K[x; \theta]/(x^n - b)$ is a sfield if and only if it has no divisors of zero [8, p. 158]. $x^n - b$ reducible implies that $(x^n - b)$ is not prime so (2) implies (1). If $K[x; \theta]/(x^n - b)$ has no divisors of zero then $x^n - b$ is prime so (3) implies (2). If $x^n - b$ is irreducible and $f(x)$ is not in $(x^n - b)$, then there are elements $f_1(x)$ and $g_1(x)$ in $K[x; \theta]$ such that $f(x)f_1(x) + (x^n - b)g_1(x) = 1$ (see [11]). Thus $f(x)$ has a right inverse (mod $(x^n - b)$) so (1) implies (3).

Since $K[x; \theta]$ is a principal ideal ring (see [11]), certainly $(x^n - b)$ prime implies that $(x^n - b)$ is maximal.

Let ϵ be a primitive 4th root of unity and x a transcendental over Q , the rational numbers. Let $F = Q(\epsilon, x)$, the field obtained by adjoining ϵ and x , and let θ be the automorphism of F determined by $\epsilon \mapsto \epsilon$ and $x \mapsto \epsilon x$. Consider the ring $F[y; \theta]$ and the ideal $(y^4 - x^4)$. Direct calculation verifies that $(y^4 - x^4)$ is maximal but not prime.

LEMMA 7. *With the same hypothesis as in Lemma 6, $b \neq \theta^{n-1}(a)\theta^{n-2}(a) \cdots \theta(a)a$ for all a in K is a necessary condition for $x^n - b$ to be irreducible polynomial in $K[x; \theta]$. For $n = 2$ or 3 it is also sufficient.*

PROOF. Let $N(a) = \theta^{n-1}(a) \cdots \theta(a) \cdot a$. If $b = N(a)$, then

$$\begin{aligned} x^n - b &= (x^{n-1} + \theta^{n-1}(a)x^{n-2} + \cdots \\ &\quad + N(a)\theta(a)^{-1}a^{-1}x + N(a)a^{-1}) \cdot (x - a). \end{aligned}$$

Thus the condition is necessary. If $n = 2$ and $x^2 - b = (x - c)(x - a)$ for a and c in K , then $ca = -b$ and $-c - \theta(a) = 0$. Thus $b = \theta(a)a$. If $n = 3$ and $x^3 - b = (x^2 + dx + c)(x - a)$ for a , b and c in K , then just in the case $n = 2$ one can verify that $b = \theta^2(a)\theta(a)a$.

With these lemmas it is easily seen that Theorem 3 can be modified to give

THEOREM 4. *Let G be an ascending solvable group with an ascending normal series $\{H_i\}_{i=0}^{\alpha}$ with $H_{i+1}/H_i = \langle x_{i+1} \cdot H_i \rangle$. Let n_{i+1} be the order of x_{i+1} mod H_i and let θ_i be the automorphism of $K[H_i]$ induced by x_{i+1} . If $n_{i+1} = 2, 3$ or ∞ for each i , then $K[G]$ has a quotient sfield if and only if for each i that n_{i+1} is finite, $x_{i+1}^{n_{i+1}} \neq \theta_i^{n_{i+1}-1}(d) \cdots \theta_i(d) \cdot d$ for all d in $K(H_i)$.*

The theory developed in Theorem 4 would be included in the results of [5] if the group algebra of an ascending solvable group G having no divisors of zero implies that G has a normal series such

that every factor is infinite cyclic. This is not true. Consider the group $G = \langle b_1, b_2, c \mid c^{-1}b_i c = b_i^{-1} \ (1 \leq i \leq 2), b_1^{-1}b_2^{-1}b_1b_2 = c^4 \rangle$. G does not have a normal series such that each factor is infinite cyclic (see [4]) but direct calculation verifies that it satisfies Theorem 4.

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UNIVERSITY OF ILLINOIS