ABSOLUTE SUMMABILITY OF A SERIES ASSOCIATED WITH A FOURIER SERIES

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1. Let $f(t)$ be a periodic function with period 2π , integrable (L) over $(-\pi, \pi)$ and

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t).$$

We write

$$\begin{aligned} \phi(t) &= \{f(\chi + t) + f(\chi - t)\}/2; & \phi^*(t) &= \phi(t) - s; \\ A_n^*(t) &= (s_n - s)/n, & n &= 1, 2, \dots, \end{aligned}$$

where $s_n = \sum_{k=0}^n A_k(t)$ and s is an appropriate number independent of n . K , denotes a positive constant, not necessarily the same at each occurrence.

If $\{p_n\}$ is a given sequence of constants, real or complex, such that $P_n = p_0 + p_1 + \dots + p_n \neq 0$, $P_{-1} = 0$, then

$$t_n = \sum_{k=0}^n P_k a_{n-k} / P_n,$$

defines the sequence $\{t_n\}$ of (N, p_n) means of $\sum_n a_n$. The series $\sum_n a_n$ is said to be summable $|N, p_n|$, if $\{t_n\} \in BV$, i.e., $\sum_n |t_n - t_{n-1}| \leq K$.

The (C) -summability of the series $\sum_{n=1}^{\infty} A_n^*(t)$ has been discussed in detail by Hardy and Littlewood [3], while a necessary and sufficient condition for its convergence is given by Zygmund [7, p. 61]. Concerning the $|C|$ -summability of the series $\sum_{n=1}^{\infty} A_n^*(t)$, Mohanty and Mohapatra [4, Theorem 3] have proved the following.

THEOREM A. *If $|\phi^*(t)|/t$ is integrable in $(0, \pi)$, then $\sum_{n=1}^{\infty} A_n^*(x)$ is summable $|C, \delta|$, $\delta > 0$.*

In the present paper we study the $|N, p_n|$ summability of the series $\sum_{n=1}^{\infty} A_n^*(x)$. We prove

THEOREM. *If $|\phi^*(t)|/t$ is integrable in $(0, \pi)$ and $\{p_n\}$ is a positive sequence such that $\{(n+1)p_n/P_n\} \in BV$, $\{P_{n-1}^{-1} \sum_{k=0}^n P_k/(k+1)\} \in BV$, then $\sum_{n=1}^{\infty} A_n^*(x)$ is summable $|N, p_n|$.*

It may be observed that Theorem A follows as a special case of our

Received by the editors October 8, 1968.

theorem since $|N, p_n|$ summability, when

$$p_n = \binom{n + \delta - 1}{\delta - 1} = \Gamma(n + \delta) / \{\Gamma(n + 1)\Gamma(\delta)\}; \quad \delta > 0,$$

is the same as $|C, \delta|$ summability.

2. We need the following lemma.

LEMMA. If $\{p_n\}$ satisfies the hypotheses of the Theorem, then uniformly in $0 < t \leq \pi$

$$\sum_{n=1}^{\infty} |g(n, t)| = \sum_{n=1}^{\infty} \left| \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - p_n P_k) \frac{\sin(n - k + \frac{1}{2})t}{n - k} \right| \leq K.$$

PROOF. The Lemma follows from the proof of $\Sigma \leq K$ in [1, p. 168], when we observe that its proof remains valid even if we replace $\sin(n - k)t$ in the expression for Σ , by $\sin(n - k + \frac{1}{2})t$.

3. **Proof of the Theorem.** For $\sum_n A^*(x)$ we have

$$\begin{aligned} t_n - t_{n-1} &= \sum_{k=0}^n \left(\frac{P_k}{P_n} - \frac{P_{k-1}}{P_{n-1}} \right) A_{n-k}^*(x) \\ &= \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - p_n P_k) A_{n-k}^*(x), \end{aligned}$$

and [6, §13.2],

$$A_n^*(x) = \frac{1}{n\pi} \int_0^\pi \phi^*(t) \{ \sin(n + \frac{1}{2})t / \sin(t/2) \} dt.$$

Thus, using the notation introduced in the Lemma, we have

$$\sum_n |t_n - t_{n-1}| \leq \int_0^\pi \left\{ \sum_n |g(n, t)| \right\} \frac{|\phi^*(t)|}{t} \frac{t}{\sin(t/2)} dt \leq K,$$

by virtue of the Lemma and the hypothesis that $|\phi^*(t)|/t$ is integrable in $(0, \pi)$. This proves the Theorem.

REMARKS. The fact that it is possible to replace the hypothesis: " $\{p_n\}$ is a positive sequence" in the Lemma by the lighter hypothesis: " $\{p_n\}$ is any sequence and $P_n^* = \sum_{k=1}^n |p_k| = O(|P_n|)$," leading to a more general version of our Theorem, follows from a very recent result of the present author [2], which shows that the hypothesis: $\sum_{k=1}^n k |p_k - p_{k-1}| = O(|P_n|)$, used by Si-Lei in [5] follows from the other hypotheses of his result [5, Theorem 1].

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