ON HILLE'S FIRST EXPONENTIAL FORMULA

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1. Introduction. E. Hille [4] and [5, p. 312] proved that for a strongly continuous semigroup of operators on a Banach space X into itself, $\{T(t); 0 \leq t < \infty\}$, the relation

$$(1.1) \quad \lim_{\tau \rightarrow 0} \|\exp(tA_\tau)f - T(t)f\| = 0$$

holds uniformly for $t \in [0, b]$, $b < \infty$ where $A_\tau = (T(\tau) - I)/\tau$. The above theorem is known as Hille's first exponential formula. L. C. Hsu [6] estimated $\exp(tA_\tau)f - T(t)f$ by:

$$(1.2) \quad \|\exp(tA_\tau)f - T(t)f\| \leq w_L(\tau^{1/3}; T(\cdot)f) + K\tau^{1/3}\|f\|,$$

where $t \in [0, b]$, $L > b$, and $w_L(\delta; T(\cdot)f)$, the rectified modulus of continuity in $[0, L]$, defined by:

$$(1.3) \quad w_L(\delta; T(\cdot)f) = \{\sup\|T(t)f - T(s)f\|; |t - s| < \delta, 0 \leq t, s < L\}.$$

Butzer and Berens [2, p. 20] mentioned that the question whether (1.2) remains valid if we substitute γ , $\gamma > \frac{1}{3}$, in place of $\frac{1}{3}$ remained undecided. The question will be answered here positively for $\gamma < \frac{1}{2}$, negatively for $\gamma > \frac{1}{2}$ and will remain open for $\gamma = \frac{1}{2}$.

2. An estimate for the first exponential formula. To state our main result we have to define the modulus of continuity of $T(t)f$ at t_0 , $t_0 \geq 0$, by:

$$(2.1) \quad w_*(\delta; T(\cdot)f, t_0) = \{\sup\|T(t)f - T(t_0)f\|, |t - t_0| < \delta, 0 \leq t\}.$$

THEOREM 2.1. Suppose $\{T(t); 0 \leq t < \infty\}$ is a strongly continuous (in $t \geq 0$) semigroup, then for $0 < \gamma < \frac{1}{2}$ and any t

$$(2.2) \quad \|\exp(tA_\tau)f - T(t)f\| \leq w_*(\tau^\gamma; T(\cdot)f, t) + K(t)e^{-(\tau)^{-\eta}}\|f\|$$

where $K(t)$ is independent of τ , $K(t)$ is bounded in any interval $[0, L]$ and $0 < \eta < \min(\frac{1}{3}, 1 - 2\gamma)$.

Since $w_L(\delta; T(\cdot)f) \geq \sup\{w_*(\delta; T(\cdot)f, t); 0 \leq t, t + \delta < L\}$ we have:

COROLLARY 2.2. Under the assumptions of Theorem 2.1 we have for $0 \leq t \leq b$, $\gamma < \frac{1}{2}$

$$(2.3) \quad \|\exp(tA_\tau)f - T(t)f\| \leq w_L(\tau^\gamma; T(\cdot)f, t) + K\tau^\gamma\|f\|$$

where K is independent of τ and $L > b + \tau^\gamma$.

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PROOF OF THEOREM 2.1. One can easily obtain

$$(2.4) \quad \begin{aligned} & \|\exp(tA_\tau)f - T(t)f\| \\ & \leq e^{-t/\tau} \sum_{k=0}^{\infty} \left(\frac{t}{\tau}\right)^k \frac{1}{k!} \|T(k\tau)f - T(t)f\| \equiv \Sigma' + \Sigma'' \end{aligned}$$

where Σ' and Σ'' are the sums on all $k \geq 0$ satisfying $|k - t/\tau| \leq \tau^{\gamma-1}$ and $|k - t/\tau| \geq \tau^{\gamma-1}$ respectively. Using (2.1) one has

$$\begin{aligned} \Sigma' & \leq e^{-t/\tau} w_*(\tau^\gamma; T(\cdot)f, t) \sum_{k=1}^{\infty} \left(\frac{t}{\tau}\right)^k \frac{1}{k!} = w_*(\tau^\gamma; T(\cdot)f, t). \\ \Sigma'' & \leq e^{-t/\tau} \Sigma^* \left(\frac{t}{\tau}\right)^k \frac{1}{k!} \|T(k\tau)f\| + e^{-t/\tau} \Sigma^{**} \left(\frac{t}{\tau}\right)^k \frac{1}{k!} \|T(t)f\| \end{aligned}$$

where Σ^* and Σ^{**} are sums on $k \geq 0$, $|k - t/\tau| > \tau^{\gamma-1}$.

Following [2, p. 20] we have $\|T(t)\| \leq M$ for $0 \leq t \leq 1$ and $\|T(t)\| \leq M^{1+t}$ for $0 \leq t < \infty$. Therefore

$$\begin{aligned} \Sigma'' & \leq M\|f\| \exp\left(-\frac{t}{\tau} + \frac{t}{\tau} M^\tau\right) \left[e^{-tM^\tau/\tau} \Sigma^* \left(\frac{tM^\tau}{\tau}\right)^k \frac{1}{k!} \right] \\ & \quad + M^{1+t}\|f\| e^{-t/\tau} \Sigma^{**} \left(\frac{t}{\tau}\right)^k \frac{1}{k!} \\ & \equiv I_1 + I_2. \end{aligned}$$

Formula (1.2.6) of [2, p. 19], that is $|tM^\tau/\tau - t/\tau| \leq t(M-1)$, yields for τ small enough:

- (a) $|k - tM^\tau/\tau| > \tau^{\gamma-1}/2$ implies $|k - t/\tau| > \tau^{\gamma-1}$,
- (b) $\exp(-t/\tau + tM^\tau/\tau) \leq e^{t(M-1)}$.

Therefore

$$I_1 \leq M\|f\| e^{t(M-1)} e^{-tM^\tau/\tau} \sum_{|k - tM^\tau/\tau| > 1/\tau^{\gamma-1}/2} \left(\frac{tM^\tau}{\tau}\right)^k \frac{1}{k!}.$$

A well-known result (see [3, pp. 200–203, Theorem 137 (9.1.6)]), yields

$$e^{-x} \sum_{|k-x| > x\zeta} \frac{x^k}{k!} = O(e^{-x^\eta})$$

for $0 < \eta < 2\zeta - 1$ as $x \rightarrow \infty$ ($\frac{1}{2} < \zeta < \frac{3}{2}$). Substitute $x = tM^\tau/\tau$ and $x^\eta < \tau^{\gamma-1}/2$ where the latter is valid if $2t^\zeta M^{\tau\zeta} < \tau^{\gamma-1+\zeta}$ which will be satisfied for small enough τ if we choose ζ such that for a fixed ϵ ,

$\epsilon > 0$, and $1 - \gamma - \epsilon > \frac{1}{2}$ we have $\zeta \leq 1 - \gamma - \epsilon$, (there exists such a ζ satisfying also $\frac{1}{2} < \zeta < \frac{2}{3}$). Therefore

$$(2.5) \quad I_1 \leq M \|f\| e^{t(M-1)K_1 e^{-\tau^{-\eta}}} \quad 0 < \eta < \min(1/3, 1 - 2\gamma).$$

(The elimination of the ϵ and the constant in the exponent is taken care of by the strict inequality $0 < \eta < \min(\frac{1}{3}, 1 - 2\gamma)$.) I_2 can be estimated similarly.

3. Further remarks. A slight generalization of Butzer's result [1, p. 119] and [2, p. 21] in which both the smoothness requirement and the estimation are local is possible.

THEOREM 3.1. *Under the assumptions of Theorem 2.1 and requiring for some $t > 0$ $w_*(\delta, T(\cdot)f, t) \leq K_1 \delta^\alpha$ for $0 < \delta \leq \delta_1$, we have*

$$(3.1) \quad \|\exp(tA_\tau)f - T(t)f\| \leq K_1 t^{\alpha/2} \tau^{\alpha/2} + O(\exp(-K_2 \tau^{-1})) \quad \tau \rightarrow 0.$$

PROOF. As in Theorem 2.1 we can write (2.4) taking here $\sum'$ and $\sum''$ to be the sums on $|k - t/\tau| \leq \delta_1$ and $|k - t/\tau| \geq \delta_1$ respectively. One can estimate $\sum'$ following Butzer [2, p. 21] and

$$\sum'' = O(\exp - K_2 \tau^{-1})$$

using (9.1.4) of Theorem 137 [3, p. 200].

Butzer's result [2, p. 21] followed by the above result shows that under an additional requirement on the modulus of continuity estimations (2.2) and (2.3), with $\gamma = \frac{1}{2}$ are valid. We shall show now by a counterexample that in general one would not be able to obtain (2.3) with $\gamma > \frac{1}{2}$.

EXAMPLE 3.2. Let X be the Banach space of uniformly continuous bounded functions on $(-\infty, \infty)$ with the norm $\|f\| = \sup_x |f(x)|$, let $T(t)f(x) = f(x+t)$ and consider $\phi \in X$, $\phi(x) = 1 - |x|$ $|x| \leq 1$, $\phi(x) = 0$ $|x| > 1$. Obviously $w(\delta, T(\cdot)\phi) = \delta$ for any $0 < \delta < 1$. We shall show that for any t $\|\exp(tA_\tau)\phi - T(t)\phi\| \geq K\tau^{1/2}$ with some $K > 0$. We have

$$\begin{aligned} & \|\exp(tA_\tau)\phi(x) - T(t)\phi(x)\| \\ &= \left\| e^{-t/\tau} \sum_{k=0}^{\infty} \left(\frac{t}{\tau}\right)^k \frac{1}{k!} (T(k\tau)\phi(x) - T(t)\phi(x)) \right\| \\ &\geq \left\| e^{-t/\tau} \sum' \left(\frac{t}{\tau}\right)^k \frac{1}{k!} (\phi(k\tau + x) - \phi(t + x)) \right\| \\ &\quad - \left\| e^{-t/\tau} \sum'' \left(\frac{t}{\tau}\right)^k \frac{1}{k!} T(k\tau)\phi(x) - T(t)\phi(x) \right\| \\ &\equiv I_1 - I_2 \end{aligned}$$

where $\sum'$ and $\sum''$ are the sums on $|k-t/\tau| \leq \delta < 1$ and $|k-t/\tau| > \delta$ ($k \geq 0$) respectively. Following Theorem 3.1, $I_2 = O(\exp(-K_2\tau^{-1}))$. To estimate I_1 from below it is enough that at some point this estimation is right (since it is sup type norm). We choose $x = -t$, for $|k\tau - t| < \tau\delta < 1$ $\phi(k\tau - t) - \phi(0) = 1 - |k\tau - t| - 1 = -|k\tau - t|$.

$$\begin{aligned} I_1 &= \left| -\sum' \left(\frac{k}{\tau} \right)^k e^{-k/\tau} \frac{1}{k!} |k\tau - t| \right| \\ &\geq \left| \sum_{k \in S} \left(\frac{k}{\tau} \right)^k e^{-k/\tau} \frac{1}{k!} (k\tau - t) \right| \end{aligned}$$

where

$$S = \left\{ k; \tau^{-1/2} < k - \frac{t}{\tau} < 2\tau^{-1/2} \right\}.$$

Using (9.1.8) of Theorem 137 of [3, p. 200] we obtain for $k \in S$

$$(3.2) \quad e^{-t/\tau} \left(\frac{t}{\tau} \right)^k \frac{1}{k!} \geq L\tau^{1/2} \quad L > 0.$$

Hence

$$I_1 \geq \tau^{-1/2} \cdot L\tau^{1/2} \cdot \max_{k \in S} |k\tau - t| / 2 \geq L\tau^{1/2}.$$

REMARK 3.3. When the modulus of continuity is known at $t_0 > 0$ then for $t > t_0$ we have the following estimation:

$$w_*(\delta, T(\cdot)f, t) \leq M e^{w(t-t_0)} w_*(\delta, T(\cdot)f, t_0)$$

since

$$\sup_{|s-t| \leq \delta} \|T(s)f - T(t)f\| = \sup_{|s_1-t_0| \leq \delta} \|T(t-t_0)[T(s_1)f - T(t_0)f]\|.$$

However the connection between the modulus of continuity at different points does not nullify the usefulness of the local approach as the following example will show. Let X be the bounded uniformly continuous function with the sup norm on $(0, \infty)$, $T(t)f(x) = f(x+t)$. Choose $\phi(x) = 0$, $0 \leq x \leq 1$; $\phi(x) = |x-1|^{1/2}$, $1 < x \leq 2$; $\phi(x) = 3-x$, $2 < x \leq 3$; and $\phi(x) = 0$, $x > 3$. Obviously for $t \leq 1$, $w_*(\delta, T(\cdot)\phi, t) \geq \delta^{1/2}$ while for $t \geq 2$, $w_*(\delta, T(\cdot)\phi, t) \leq \delta$.

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