THE FREE METABELIAN GROUP OF EXPONENT p^2

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Introduction. For $n \geq 2$ and p a prime, let M_{n,p^2} denote the free metabelian group of exponent p^2 on n generators. The precise nilpotency class of $M_{n,4}$ ($n \geq 2$) was established in Gupta and Tobin [2], where it was also shown that $|M_{2,4}| = 2^{10}$ and $|M_{3,4}| = 2^{34}$. In a recent paper [1], Bachmuth and Mochizuki have shown that the class of M_{2,p^2} (p odd) is precisely $2(p^2 - p)$. In another paper (to appear) Bachmuth, Heilbronn and Mochizuki have shown that for $3 \leq k \leq p+1$, the class of M_{k,p^2} is $2(p^2 - p)$ or $2(p^2 - p) + 1$. In this paper we complete the discussion for the case $k \geq p+2$.

THEOREM. *If $n \geq p+2$, the class of M_{n,p^2} is precisely $n(p-1) + (p-1)^2$.*

Proof of the theorem. For $n \geq p+2$, Theorem 3 of Gupta, Newman and Tobin [3] gives $n(p-1) + (p-1)^2$ as an upper bound for the nilpotency class of M_{n,p^2} . In this note we obtain $n(p-1) + (p-1)^2$ as a lower bound by exhibiting a metabelian group G on n generators which is of exponent p^2 and the $(n(p-1) + (p-1)^2)$ th term of whose lower central series is nontrivial.

Let $A = gp\{a_1, \dots, a_n \mid n \geq 2\}$; $a_1^{p^2}, a_i^p$ ($i = 2, \dots, n$), $a_i a_j = a_j a_i$ ($i, j = 1, \dots, n$), and let $Z_p A$ be the group ring of A over integers mod p . Let J denote the ideal of $Z_p A$ generated by all polynomials $1 + a + \dots + a^{p^2-1}$ for $a \in A$ and let $\hat{J}$ denote the ideal of $Z_p A$ generated by $(1 - a_1)^{p^2-p}$. If $a \in gp\{a_2, \dots, a_n\}$, then

$$1 + a + \dots + a^{p^2-1} = p(1 + a + \dots + a^{p-1}) = 0.$$

Thus each generating element of J is of the form $1 + a + \dots + a^{p^2-1}$, where $a = a_1^s a^*$ for some $s \in \{1, \dots, p^2-1\}$ and $a^* \in gp\{a_2, \dots, a_n\}$. On the other hand

$$\begin{aligned} 1 + a + \dots + a^{p^2-1} &= (1 + a + \dots + a^{p-1})(1 + a^p + \dots + a^{p(p-1)}) \\ &= \rho \cdot (1 + a_1^{sp} + \dots + a_1^{sp(p-1)}) \quad (\text{where } \rho \in Z_p A) \\ &= \rho(1 - a_1^s)^{p(p-1)} \in \hat{J}; \end{aligned}$$

so that $J \subseteq \hat{J}$, and we conclude that

$$(1) \quad (1 - a_1)^{p^2-p-1} (1 - a_2)^{p-1} \dots (1 - a_n)^{p-1} \notin J.$$

Let $\{\lambda_1, \dots, \lambda_n\}$ be a set of independent and commuting indeterminates which also commute with every element of $Z_p A/J$ and form

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the polynomial ring $R = Z_p A / J[\lambda_1, \dots, \lambda_n]$ with coefficients in $Z_p A / J$. For $i = 1, \dots, n$, let a_i denote the 2×2 matrix

$$\begin{pmatrix} 1 & \lambda_i \\ 0 & a_i \end{pmatrix}$$

over R and let G be the group generated by a_i 's under matrix multiplication. By a well-known result of Magnus [4] G is metabelian, and it is easily seen to be of exponent p^2 .

We note that if

$$a = [a_1, a_2, \dots, a_2, \dots, a_n, \dots, a_n, a_1, \dots, a_1],$$

where $a_2, \dots, a_n$ repeat $p-1$ times and a_1 repeats p^2-p-1 times, then

$$a = \begin{pmatrix} 1 & \alpha_1 \lambda_1 + \alpha_2 \lambda_2 \\ 0 & 1 \end{pmatrix},$$

where $\alpha_1 = -(1-a_2)^{p-1} \dots (1-a_n)^{p-1} (1-a_1)^{p^2-p-1}$ and $\alpha_2 = (1-a_2)^{p-2} \dots (1-a_n)^{p-2} (1-a_1)^{p^2-p}$.

Since λ_1, λ_2 are independent,

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

if and only if both α_1 and α_2 are in J . Since $\alpha_1 \notin J$ by (1), we conclude that a is not the identity element of G . Thus the $(n(p-1) + (p-1))$ th term of the lower central series of G is nontrivial and so G has class precisely $n(p-1) + (p-1)^2$.

ADDED IN PROOF. Dr. M. F. Newman has recently proved (written communication) that the bounds obtained in [3] are best possible. Thus our theorem also follows from his general results.

REFERENCES

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