ISOMORPHISMS GENERATED BY FUNDAMENTAL AND TOTAL SETS

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A biorthogonal system $\{x_i, f_i\}_{i \in I}$ in a linear topological duality (X, X^*) is a Generalized Basis (G-Basis) if $\{f_i\}$ is total over X ; a Dual Generalized Basis (DG-Basis) if $\{x_i\}$ is fundamental in X . Arsove and Edwards [1] proved that the existence of similar G-Bases in Fréchet spaces X and Y implies that X and Y are isomorphic (linearly homeomorphic). Davis [2] noted that if $\{x_i, f_i\}$ is a DG-Basis for X , then $\{f_i, x_i\}$ is a G-Basis for X^* if X^* is endowed with any topology stronger than $w(X^*, X)$. This observation and the isomorphism theorem of Arsove and Edwards allowed him to prove that the existence of *similar DG-Bases in Banach spaces X and Y implies that X and Y are isomorphic.

The definitions of similarity and *similarity can be extended to structures more general than G- and DG-Bases (Definitions 1 and 2). Similarity implies the existence of an algebraic isomorphism with closed graph between the spaces (Theorem 1), and thus the Arsove-Edwards result is extended to fully complete barreled spaces. The assumption of *similarity yields an isomorphism theorem for a class of spaces which includes Fréchet spaces (Theorem 2).

The notation and terminology of [5] is used throughout.

I. The isomorphism theorems.

DEFINITION 1. Suppose $\{f_i\}_{i \in I}$ and $\{g_i\}_{i \in I}$ are total sets of continuous linear functionals on linear topological spaces X and Y respectively. $\{f_i\}$ and $\{g_i\}$ are *similar* iff $\{\{f_i(x)\}_{i \in I} : x \in X\} = \{\{g_i(y)\}_{i \in I} : y \in Y\}$.

THEOREM 1. Suppose $\{f_i\}_{i \in I}$ and $\{g_i\}_{i \in I}$ are similar total sets of continuous linear functionals on (X, T) and (Y, S) , respectively. Then there is an algebraic isomorphism, L , from X onto Y such that $f_i = g_i \circ L$ for each $i \in I$. L is a $w(X, \{f_i\}) - w(Y, \{g_i\})$ homeomorphism, and hence has closed graph in $X \times Y$ with the product topology $T \times S$.

Assume that X and Y satisfy the conditions of Theorem 1. If there is a closed graph theorem for linear maps from X to Y and Y to X , then L is an isomorphism. This will be the case, for example, if X and Y are barreled and fully complete [5, p. 179, 18.J], or the induc-

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tive limit of a sequence of fully complete Baire spaces [6, p. 124]. The referee notes that recent works of de Wilde on closed graph theorems give still more applications of Theorem 1.

DEFINITION 2. Suppose $\{x_i\}_{i \in I}$ and $\{y_i\}_{i \in I}$ are fundamental linearly independent subsets of linear topological spaces X and Y respectively. $\{x_i\}$ and $\{y_i\}$ are **similar* iff $\{\{f(x_i)\}_{i \in I} : f \in X^*\} = \{\{g(y_i)\}_{i \in I} : g \in Y^*\}$.

A subset, A , of a linear topological space, X , is called sequentially fundamental provided each element of X is the limit of a sequence of linear combinations of elements of A .

THEOREM 2. Suppose (X, T) and (Y, S) are Mackey linear topological spaces, and that $\{x_i\}_{i \in I}$ and $\{y_i\}_{i \in I}$ are **similar fundamental linearly independent subsets of X and Y respectively. Suppose either that*

(1) $m(\text{sp}\{x_i\}, X^*)$ is the relativisation of T to $\text{sp}\{x_i\}$ and $m(\text{sp}\{y_i\}, Y^*)$ is the relativisation of S to $\text{sp}\{y_i\}$; and

(a) X and Y are both complete, or

(b) X and Y are both sequentially complete and $\{x_i\}$ and $\{y_i\}$ are both sequentially fundamental; or

(2) $\{x_i\}$ and $\{y_i\}$ are both weakly sequentially fundamental and both X and Y are weakly sequentially complete.

Then there is a T — S isomorphism, $\bar{L}$, of X onto Y such that $\bar{L}(x_i) = y_i$, for each $i \in I$.

PROOF. Define $L: \text{sp}(\{x_i\}) \rightarrow \text{sp}(\{y_i\})$ by $L(\sum_{i \in F} t_i x_i) = \sum_{i \in F} t_i y_i$, for all finite subsets, F , of I and scalars t_i . L is well defined because $\{x_i\}$ is linearly independent; one-to-one, because $\{y_i\}$ is linearly independent. L is clearly onto and linear, so that L is an algebraic isomorphism of $\text{sp}\{x_i\}$ onto $\text{sp}\{y_i\}$. We show that L is weakly continuous. Let $\{z_d\}$ be a net in $\text{sp}\{x_i\}$ which converges weakly to 0. Let $g \in Y^*$. By the **similarity* of $\{x_i\}$ and $\{y_i\}$, there is $f \in X^*$ such that for all $i \in I$, $f(x_i) = g(y_i)$. It follows that for all d , $g(L(z_d)) = f(z_d)$. By the assumption on $\{z_d\}$, $\{f(z_d)\}$ converges to 0. Thus L is weakly continuous. Similarly, L^{-1} is weakly continuous, so that L is a weak isomorphism.

Case 1. Since relativisations of weak topologies are weak [5, p. 160, 17.13], it follows from [5, p. 202, 21.4] that L is a Mackey isomorphism, hence by (1) L is a T — S isomorphism. Thus $\bar{L}$, the closure of L in $(X, T) \times (Y, S)$, is a T — S isomorphism. Either (a) or (b) guarantees that $\bar{L}$ has domain X and range Y .

Case 2. Let $\bar{L}$ be the weak closure of L in $X \times Y$. $\bar{L}$ is a weak iso-

morphism since L is, so that by [5, p. 202, 21.4] $\bar{L}$ is a T - S isomorphism. (2) guarantees that $\bar{L}$ has domain X and range Y .

Since subspaces of Fréchet spaces with the relative topology are Mackey [5, p. 210, 22.3], Case 1 of Theorem 2 applies to Fréchet spaces. Case 2 applies to weak Schauder Bases in weakly sequentially complete spaces.

II. Markushevich bases and Schauder bases. $\{x_i, f_i\}_{i \in I}$ is a *Markushevich Basis* (M-Basis) for X iff $\{x_i, f_i\}$ is both a G- and DG-Basis for X . A Schauder Basis is an M-Basis. Schauder Bases $\{x_i, f_i\}_{i=1}^\infty$ and $\{y_i, g_i\}_{i=1}^\infty$ are similar iff for every sequence $\{t_i\}_{i=1}^\infty$ of scalars, $\sum_{i=1}^\infty t_i x_i$ exists iff $\sum_{i=1}^\infty t_i y_i$ exists. Corollary 1.1 and Theorem 2 yield the conclusion that M-Bases in Fréchet spaces are similar iff they are *similar. The relationship between similarity and *similarity is more complicated in the general setting, even in the case of Schauder Bases.

EXAMPLE 1. Let C be an infinite dimensional Banach space with Schauder Basis $\{x_i, f_i\}_{i=1}^\infty$. Let D be a proper subspace of C such that D contains each x_i and D is of the second category [5, p. 95, 10.B]. Then C and D are both barreled and $\{x_i, f_i\}$ is a Schauder Basis for D which is *similar to the Schauder Basis $\{x_i, f_i\}$ for C . Since D is a proper subspace of C , $\{x_i, f_i\}$ in C cannot be similar to $\{x_i, f_i\}$ in D .

EXAMPLE 2. Consider c_0 with the usual $\{e_i, e'_i\}_{i=1}^\infty$ basis. Then $\{e'_i, e_i\}_{i=1}^\infty$ is a Schauder Basis for $l_1 (= c_0^*)$ with either the norm or the $m(l_1, c_0)$ topology. The two $\{e'_i, e_i\}$ bases are easily seen to be similar but not *similar. Note that both topologies on l_1 are Mackey and fully complete, but the $m(l_1, c_0)$ topology is not barreled.

If $\{x_i, f_i\}_{i=1}^\infty$ is a Schauder Basis for a barreled space X , then a linear functional f is continuous iff $f(x) = \sum_{i=1}^\infty f_i(x)f(x_i)$, for all $x \in X$ [4]. Thus, similar Schauder Bases in barreled spaces are *similar. Dubinsky and Retherford [3] have noted that if $\{x_i, f_i\}_{i=1}^\infty$ and $\{y_i, g_i\}_{i=1}^\infty$ are similar and *similar Schauder Bases (or even weak Schauder Bases), then the map $x_i \rightarrow y_i$ induces a Mackey isomorphism. In case $\{x_i, f_i\}$ and $\{y_i, g_i\}$ are M-Bases satisfying certain additional conditions, Theorem 3 yields the same conclusion.

DEFINITION 3. An M-Basis $\{x_i, f_i\}_{i \in I}$ for X has *property α* iff $\{f_i\}$ is fundamental in X^* when X^* is endowed with the strong topology $s(X^*, X)$.

An M-Basis $\{x_i, f_i\}$ for X has property α iff $\{f_i, x_i\}$ is an M-Basis for X^* with topology $s(X^*, X)$. Since $\{f_i, x_i\}$ is always an M-Basis for X^* with topology $w(X^*, X)$ and hence with topology $m(X^*, X)$, semireflexivity of X implies that $\{x_i, f_i\}$ has property α .

DEFINITION 4. A set $\{x_i\}_{i \in I}$ in X is *bounded fundamental* iff for each $x \in X$ there is a bounded net $\{z_d\}$ in $\text{sp}(\{x_i\})$ such that $\{z_d\}$ converges to x .

Since convergent sequences are bounded, a sequentially fundamental set is bounded fundamental.

THEOREM 3. Suppose $\{x_i, f_i\}_{i \in I}$ and $\{y_i, g_i\}_{i \in I}$ are similar and *similar M-Bases for Mackey spaces (X, T) and (Y, S) , respectively. Suppose further that $\{x_i\}$ and $\{y_i\}$ are weakly bounded fundamental in X and Y , respectively, and that both $\{x_i, f_i\}$ and $\{y_i, g_i\}$ have property α . Then there is an isomorphism, $\bar{L}$, of X onto Y such that $\bar{L}(x_i) = y_i$, for all $i \in I$.

PROOF. Let $L: \text{sp}(\{x_i\}) \rightarrow \text{sp}(\{y_i\})$ be defined as in Theorem 2. L is a weak isomorphism, hence $\bar{L}$, the weak closure of L in $X \times Y$, is a weak isomorphism. We show that $\bar{L}$ has domain X . Let $x \in X$. Since $\{x_i\}$ is weakly bounded fundamental, there is a bounded net $\{z_d: d \in D\}$ in $\text{sp}\{x_i\}$ which converges weakly to x . Since $\{x_i\}$ and $\{y_i\}$ are similar, there is a $y \in Y$ such that for all $i \in I$, $f_i(x) = g_i(y)$. It follows from the definition of L that $f_i(z_d) = g_i(L(z_d))$, for all $i \in I$ and $d \in D$. We show that $\{L(z_d): d \in D\}$ converges weakly to y . Let $g \in Y^*$ and let $\epsilon > 0$. L is weakly continuous and $\{z_d\}$ is bounded, hence $\{L(z_d): d \in D\} \cup \{y\}$ is bounded. Since $\{y_i, g_i\}$ has property α , there is a finite subset F of I and scalars t_i such that $|\sum_{i \in F} t_i g_i(y') - g(y')| < \epsilon/3$, for all $y' \in \{L(z_d): d \in D\} \cup \{y\}$. Since $\{z_d\}$ converges weakly to x , there is $e \in D$ such that for all $d \geq e$, $|\sum_{i \in F} t_i f_i(x - z_d)| < \epsilon/3$. Then for all $d \geq e$,

$$\begin{aligned} |g(y) - g(L(z_d))| &\leq \left| g(y) - \sum_{i \in F} t_i g_i(y) \right| \\ &\quad + \left| \sum_{i \in F} t_i g_i(y) - \sum_{i \in F} t_i g_i(L(z_d)) \right| \\ &\quad + \left| \sum_{i \in F} t_i g_i(L(z_d)) - g(L(z_d)) \right| \\ &< \epsilon/3 + \left| \sum_{i \in F} t_i f_i(x) - \sum_{i \in F} t_i f_i(z_d) \right| + \epsilon/3 < \epsilon. \end{aligned}$$

Thus $\{L(z_d)\}$ converges weakly to y and hence the domain of $\bar{L}$ is X . Similarly, the range of $\bar{L}$ is Y . The desired conclusion now follows from [5, p. 202, 21.4].

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