AN INEQUALITY OF SCHUR AND AN INEQUALITY OF NEWTON

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Let E_r^1 and H_r^1 denote the elementary and complete symmetric functions of the r th order in $\alpha_1, \alpha_2, \dots, \alpha_m$ respectively.

If

$$(1.1) \quad E_r = E_r^1 / \binom{m}{r}$$

and

$$(1.2) \quad H_r = H_r^1 / \binom{m+r-1}{r}$$

then we have two well-known inequalities

$$(1.3) \quad E_r^2 \geq E_{r-1}E_{r+1}$$

and

$$(1.4) \quad H_r^2 \leq H_{r-1}H_{r+1}.$$

The inequality (1.3) is known as Newton's inequality and is true when $\alpha_1, \alpha_2, \dots, \alpha_m$ are real numbers. The inequality (1.4), given by Professor Schur, is true when $\alpha_1, \alpha_2, \dots, \alpha_m$ are nonnegative reals. In [1], Whiteley has given the following generalizations of (1.3) and (1.4): Let

$$(1.5) \quad E_r^1(K) = \sum \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_m} \alpha_1^{i_1} \alpha_2^{i_2} \dots \alpha_m^{i_m}, \quad i_1 + i_2 + \dots + i_m = r,$$

$$\lambda_i = \binom{K}{i}$$

and K any positive real number. Similarly we define,

$$(1.6) \quad H_r^1(K) = \sum \delta_{i_1} \delta_{i_2} \dots \delta_{i_m} \alpha_1^{i_1} \alpha_2^{i_2} \dots \alpha_m^{i_m},$$

$$i_1 + i_2 + \dots + i_m = r,$$

where

$$\delta_i = (-1)^i \binom{K}{i}$$

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and K any negative real number. In [1], it is proved that if

$$(1.7) \quad E_r(K) = E_r^1(K) / \binom{Km}{r},$$

$$(1.8) \quad H_r(K) = H_r^1(K) / \binom{Km}{r},$$

then

$$(1.9) \quad [E_r(K)]^2 \geq E_{r-1}(K) \cdot E_{r+1}(K),$$

$$(1.10) \quad [H_r(K)]^2 \leq H_{r-1}(K) \cdot H_{r+1}(K).$$

In this paper we prove that (1.9) and (1.10) are true for a more general type of nonsymmetric functions

$$(1.11) \quad E_r(K_1, K_2, \dots, K_m)$$

$$(1.12) \quad H_r(K_1, K_2, \dots, K_m)$$

which are defined in the following way: Let

$$(1.13) \quad E_r^1(K_1, K_2, \dots, K_m) = \sum \binom{K_1}{i_1} \binom{K_2}{i_2} \dots \binom{K_m}{i_m} \alpha_1^{i_1} \alpha_2^{i_2} \dots \alpha_m^{i_m},$$

$$i_1 + i_2 + \dots + i_m = r;$$

$$(1.14) \quad H_r^1(K_1, K_2, \dots, K_m) = \sum \binom{K_1 + i_1 - 1}{i_1} \binom{K_2 + i_2 - 1}{i_2} \dots$$

$$\cdot \binom{K_m + i_m - 1}{i_m} \alpha_1^{i_1} \alpha_2^{i_2} \dots \alpha_m^{i_m},$$

$$i_1 + i_2 + \dots + i_m = r,$$

where $K_1, K_2, \dots, K_m$ are positive real numbers. Now we define

$$(1.15) \quad E_r(K_1, K_2, \dots, K_m) = \frac{E_r^1(K_1, K_2, \dots, K_m)}{\binom{K_1 + K_2 + \dots + K_m}{r}},$$

$$(1.16) \quad H_r(K_1, K_2, \dots, K_m) = \frac{H_r^1(K_1, K_2, \dots, K_m)}{\binom{K_1 + K_2 + \dots + K_m + r - 1}{r}};$$

we note that

$$(1.17) \quad \begin{aligned} E_r(K, K, K, \dots, K) &= E_r(K) \quad \text{and} \\ H_r(K, K, K, \dots, K) &= H_r(K), \end{aligned}$$

$$(1.18) \quad E_r(1, 1, \dots, 1) = E_r \quad \text{and} \quad H_r(1, 1, \dots, 1) = H_r.$$

The generating series for (1.13) is

$$(1.19) \quad 1 + \sum E_r^1(K_1, K_2, \dots, K_m) x^r = \prod_{i=1}^m (1 + \alpha_i x)^{K_i}$$

and the generating series for (1.14) is

$$(1.20) \quad 1 + \sum H_r^1(K_1, K_2, \dots, K_m) x^r = \prod_{i=1}^m (1 - \alpha_i x)^{-K_i}$$

where $K_1, K_2, \dots, K_m$ are positive real numbers. In the proof of the theorem we make use of the classical theory of maxima and minima as in [1].

2. LEMMA 1. *If $r=1$, then for all m ,*

$$(2.1) \quad \begin{aligned} [H_r(K_1, K_2, \dots, K_m)]^2 \\ \leq H_{r-1}(K_1, K_2, \dots, K_m) H_{r+1}(K_1, K_2, \dots, K_m) \end{aligned}$$

where $\alpha_1, \alpha_2, \dots, \alpha_m$ are nonnegative reals. The inequality is strict unless all the variables are equal.

PROOF.

$$(2.2) \quad H_1(K_1, K_2, \dots, K_m) = \frac{(\sum_{i=1}^m K_i \alpha_i)}{(\sum_{i=1}^m K_i)},$$

$$(2.3) \quad H_2(K_1, K_2, \dots, K_m) = \frac{\sum_{i=1}^m K_i(K_i + 1)\alpha_i^2 + \sum_{i \neq j} 2K_i K_j \alpha_i \alpha_j}{(\sum_{i=1}^m K_i)(\sum_{i=1}^m K_i + 1)}.$$

Hence

$$(2.4) \quad \begin{aligned} H_2(K_1, K_2, \dots, K_m) - [H_1(K_1, K_2, \dots, K_m)]^2 \\ = \frac{1}{(\sum_{i=1}^m K_i)^2 (\sum_{i=1}^m K_i + 1)} \\ \cdot \left\{ \left(\sum_{i=1}^m K_i \right) \left[\sum_{i=1}^m K_i(K_i + 1)\alpha_i^2 + 2 \sum_{i \neq j} K_i K_j \alpha_i \alpha_j \right] \right. \\ \left. - \left(\sum_{i=1}^m K_i + 1 \right) \left(\sum_{i=1}^m K_i \alpha_i \right)^2 \right\}. \end{aligned}$$

Or,

$$(2.5) \quad \begin{aligned} & H_2(K_1, K_2, \dots, K_m) - [H_1(K_1, K_2, \dots, K_m)]^2 \\ &= \frac{1}{(\sum_{i=1}^m K_i)^2 (\sum_{i=1}^m K_i + 1)} \left\{ \sum_{i \neq j} K_i K_j (\alpha_i - \alpha_j)^2 \right\}. \end{aligned}$$

Hence the result.

3. LEMMA 2. If $r = 1$, then for all m ,

$$(3.1) \quad \begin{aligned} & [E_r(K_1, K_2, \dots, K_m)]^2 \\ & \geq E_{r-1}(K_1, K_2, \dots, K_m) E_{r+1}(K_1, K_2, \dots, K_m) \end{aligned}$$

where $\alpha_1, \alpha_2, \dots, \alpha_m$ are nonnegative reals. The inequality is strict unless all the variables are equal.

PROOF. Same as Lemma 2.

4. LEMMA 3. For all r , if $m = 1$, then

$$(4.1) \quad \begin{aligned} & H_{r-1}(K_1, K_2, \dots, K_m) H_{r+1}(K_1, K_2, \dots, K_m) \\ & - [H_r(K_1, K_2, \dots, K_m)]^2 = 0 \end{aligned}$$

and

$$(4.2) \quad \begin{aligned} & E_{r-1}(K_1, K_2, \dots, K_m) E_{r+1}(K_1, K_2, \dots, K_m) \\ & - [E_r(K_1, K_2, \dots, K_m)]^2 = 0. \end{aligned}$$

PROOF. Follows from (1.15) and (1.16).

5. LEMMA 4.

$$(5.1) \quad \begin{aligned} & \sum_{i=1}^m \frac{\partial}{\partial \alpha_i} H_r^1(K_1, K_2, \dots, K_m) \\ &= \left(\sum_{i=1}^m K_i + r - 1 \right) H_{r-1}(K_1, K_2, \dots, K_m), \end{aligned}$$

$$(5.2) \quad \begin{aligned} & \sum_{i=1}^m \frac{\partial}{\partial \alpha_i} E_r^1(K_1, K_2, \dots, K_m) \\ &= \left(\sum_{i=1}^m K_i - r + 1 \right) E_{r-1}(K_1, K_2, \dots, K_m). \end{aligned}$$

PROOF. By Euler's theorem on homogeneous functions we have

$$(5.3) \quad \sum_{i=1}^m \alpha_i \frac{\partial}{\partial \alpha_i} H_r^1(K_1, K_2, \dots, K_m) = r H_r^1(K_1, K_2, \dots, K_m).$$

From (1.20) and (5.3) we get (5.1).

In the same way (5.2) can be proved from (1.19).

6. THEOREM 1. *If $\alpha_1, \alpha_2, \dots, \alpha_m$ are nonnegative reals, then*

$$(6.1) \quad H_{r-1}(K_1, K_2, \dots, K_m) H_{r+1}(K_1, K_2, \dots, K_m) > [H_r(K_1, K_2, \dots, K_m)]^2$$

and

$$(6.2) \quad H_r(K_1, K_2, \dots, K_m)^{1/r} > H_{r+1}(K_1, K_2, \dots, K_m)^{1/(r+1)}$$

where $r = 1, 2, 3, \dots$. The inequality is strict unless all the variables are equal.

PROOF. Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be m variables subject to the following conditions:

$$\alpha_1 > 0, \quad \alpha_2 > 0, \dots, \alpha_m > 0$$

and

$$(6.3) \quad H_{r-1}^1(K_1, K_2, \dots, K_m) H_{r+1}^1(K_1, K_2, \dots, K_m) = 1.$$

These conditions define a closed set of points in m -dimensional space. This set of points is bounded also since the function

$$H_{r-1}^1(K_1, K_2, \dots, K_m)$$

and

$$H_{r+1}^1(K_1, K_2, \dots, K_m)$$

has positive coefficients and includes the term α_i^{2r} for $i = 1, 2, \dots, m$. Also from Lemma 1 the theorem is true for all m if $r = 1$, and by Lemma 3, the theorem is true for all r , if $m = 1$. We use double induction on m and r to prove the theorem. We shall prove that it holds for all m and r provided it holds for all pairs m, r with $m_1 < m$, and for all pairs m, r_1 with $r_1 < r$. By Euler's theorem on homogeneous functions and (6.3)

$$(6.4) \quad \sum \alpha_i \frac{\partial}{\partial \alpha_i} H_{r-1}^1(K_1, K_2, \dots, K_m) H_{r+1}^1(K_1, K_2, \dots, K_m) = 2r H_{r-1}^1(K_1, K_2, \dots, K_m) H_{r+1}^1(K_1, K_2, \dots, K_m) = 2r.$$

From (6.4) we note that the first partial derivatives cannot all vanish at the point at which $[H_r^1(K_1, K_2, \dots, K_m)]^2$ takes its maximum value, say M . Applying Lagrange's conditions at this point we have

$$\begin{aligned}
 (6.5) \quad & \frac{\partial}{\partial \alpha_i} [H_r^1(K_1, K_2, \dots, K_m)]^2 \\
 & - \lambda \frac{\partial}{\partial \alpha_i} H_{r-1}^1(K_1, K_2, \dots, K_m) H_{r+1}^1(K_1, K_2, \dots, K_m) = 0.
 \end{aligned}$$

From (6.5) we have

$$\begin{aligned}
 (6.6) \quad & 2H_r^1(K_1, K_2, \dots, K_m) \frac{\partial}{\partial \alpha_i} H_r^1(K_1, K_2, \dots, K_m) \\
 & = \lambda \left\{ H_{r-1}^1(K_1, K_2, \dots, K_m) \frac{\partial}{\partial \alpha_i} H_{r+1}^1(K_1, K_2, \dots, K_m) \right. \\
 & \quad \left. + H_{r+1}^1(K_1, K_2, \dots, K_m) \frac{\partial}{\partial \alpha_i} H_{r-1}^1(K_1, K_2, \dots, K_m) \right\}.
 \end{aligned}$$

Now multiplying each of (6.6) by α_i and adding we have

$$\begin{aligned}
 (6.7) \quad & 2H_r^1(K_1, K_2, \dots, K_m) \sum_{i=1}^m \alpha_i \frac{\partial}{\partial \alpha_i} H_r^1(K_1, K_2, \dots, K_m) \\
 & - \lambda \left\{ H_{r-1}^1(K_1, K_2, \dots, K_m) \sum_{i=1}^m \alpha_i \frac{\partial}{\partial \alpha_i} H_{r+1}^1(K_1, K_2, \dots, K_m) \right. \\
 & \quad \left. + H_{r+1}^1(K_1, K_2, \dots, K_m) \sum_{i=1}^m \alpha_i \frac{\partial}{\partial \alpha_i} H_{r-1}^1(K_1, K_2, \dots, K_m) \right\} = 0.
 \end{aligned}$$

But

$$\begin{aligned}
 (6.8) \quad & \sum_{i=1}^m \alpha_i \frac{\partial}{\partial \alpha_i} H_{r+1}^1(K_1, K_2, \dots, K_m) \\
 & = (r+1) H_{r+1}^1(K_1, K_2, \dots, K_m).
 \end{aligned}$$

Hence from (6.7) and (6.8) we have

$$\begin{aligned}
 (6.9) \quad & 2r[H_r^1(K_1, K_2, \dots, K_m)]^2 \\
 & - \lambda 2r H_{r-1}^1(K_1, K_2, \dots, K_m) H_{r+1}^1(K_1, K_2, \dots, K_m) = 0
 \end{aligned}$$

or,

$$(6.10) \quad \lambda = \frac{[H_r^1(K_1, K_2, \dots, K_m)]^2}{[H_{r-1}^1(K_1, K_2, \dots, K_m) H_{r+1}^1(K_1, K_2, \dots, K_m)]}.$$

Now using (6.6)

$$\begin{aligned}
 (6.11) \quad & 2H_r^1(K_1, K_2, \dots, K_m) \left(\sum_{i=1}^m K_i + r - 1 \right) H_{r-1}^1(K_1, K_2, \dots, K_m) \\
 &= \lambda \left\{ H_{r-1}^1(K_1, K_2, \dots, K_m) \left(\sum_{i=1}^m K_i + r \right) \right. \\
 &\quad \cdot H_r^1(K_1, K_2, \dots, K_m) + H_r^1(K_1, K_2, \dots, K_m) \\
 &\quad \cdot \left. \left(\sum_{i=1}^m K_i + r - 2 \right) H_{r-2}^1(K_1, K_2, \dots, K_m) \right\}.
 \end{aligned}$$

Now from (6.11) we have,

$$\begin{aligned}
 (6.12) \quad & 2 \left\{ \sum_{i=1}^m K_i + r - 1 \right\} - \lambda \left\{ \sum_{i=1}^m K_i + r \right\} \\
 &= \lambda \left\{ \sum_{i=1}^m K_i + r - 2 \right\} \\
 &\quad \cdot \frac{H_{r+1}^1(K_1, K_2, \dots, K_m) H_{r-2}^1(K_1, K_2, \dots, K_m)}{H_r^1(K_1, K_2, \dots, K_m) H_{r-1}^1(K_1, K_2, \dots, K_m)}.
 \end{aligned}$$

Now from (6.12) and (6.10) we have

$$\begin{aligned}
 (6.13) \quad & 2 \left\{ \sum_{i=1}^m K_i + r - 1 \right\} - \lambda \left\{ \sum_{i=1}^m K_i + r \right\} \\
 &= \left\{ \sum_{i=1}^m K_i + r - 2 \right\} \\
 &\quad \cdot \frac{H_r^1(K_1, K_2, \dots, K_m) H_{r-2}^1(K_1, K_2, \dots, K_m)}{[H_{r-1}^1(K_1, K_2, \dots, K_m)]^2}.
 \end{aligned}$$

But by induction hypothesis

$$\begin{aligned}
 (6.14) \quad & \frac{H_r^1(K_1, K_2, \dots, K_m) H_{r-2}^1(K_1, K_2, \dots, K_m)}{[H_{r-1}^1(K_1, K_2, \dots, K_m)]^2} \\
 &\geq \frac{(\sum_{i=1}^m K_i + r - 1)}{(\sum_{i=1}^m K_i + r - 2)} \frac{(r - 1)}{r}.
 \end{aligned}$$

Hence

$$2 \left(\sum_{i=1}^m K_i + r - 1 \right) - \lambda \left(\sum_{i=1}^m K_i + r \right) \\ \geq \left(\sum_{i=1}^m K_i + r - 2 \right) \frac{r-1}{r} \frac{(\sum_{i=1}^m K_i + r - 1)}{(\sum_{i=1}^m K_i + r - 2)}.$$

Therefore,

$$2 \left(\sum_{i=1}^m K_i + r - 1 \right) - \lambda \left(\sum_{i=1}^m K_i + r \right) \geq \frac{r-1}{r} \left(\sum_{i=1}^m K_i + r - 1 \right)$$

or

$$\left(\sum_{i=1}^m K_i + r - 1 \right) \frac{r+1}{r} \geq \lambda \left(\sum_{i=1}^m K_i + r \right)$$

or

$$(6.15) \quad \lambda < \frac{r+1}{r} \frac{(\sum_{i=1}^m K_i + r - 1)}{(\sum_{i=1}^m K_i + r)}.$$

In the next place we suppose that the maximum M is attained when $\alpha_{m_1+1}=0$, $\alpha_{m_1+2}=0$, $\dots$, $\alpha_m=0$, $\alpha_1 \neq 0$, $\dots$, $\alpha_{m_1} \neq 0$, ($m_1 < m$). Induction on m gives

$$(6.16) \quad \lambda = \frac{[H_r^1(K_1, K_2, \dots, K_{m_1})]^2}{H_{r-1}^1(K_1, K_2, \dots, K_{m_1}) H_{r+1}^1(K_1, K_2, \dots, K_{m_1})};$$

then

$$(6.17) \quad \lambda \leq \frac{r+1}{r} \frac{(\sum_{i=1}^{m_1} K_i + r - 1)}{(\sum_{i=1}^{m_1} K_i + r)},$$

and clearly

$$\frac{r+1}{r} \frac{\sum_{i=1}^{m_1} K_i + r - 1}{\sum_{i=1}^{m_1} K_i + r} < \frac{r+1}{r} \frac{\sum_{i=1}^m K_i + r - 1}{\sum_{i=1}^m K_i + r};$$

hence the theorem.

The second part of the theorem can be proved as in [2].

7. THEOREM 2. If $\alpha_1, \alpha_2, \dots, \alpha_m$ are nonnegative reals, then

$$(7.1) \quad [E_r(K_1, K_2, \dots, K_m)]^2 \\ > E_{r-1}(K_1, K_2, \dots, K_m) E_{r+1}(K_1, K_2, \dots, K_m)$$

and

$$E_r(K_1, K_2, \dots, K_m)^{1/r} > E_r(K_1, K_2, \dots, K_m)^{1/r+1},$$

where $r = 1, 2, \dots, r < K + 1$, where K is not integral and K denotes the smallest of the real positive numbers, $K_1, K_2, \dots, K_m$. The inequality is strict unless all the variables are equal.

PROOF. The restriction on r , makes all the terms positive, and hence the theorem can be proved in the same way as Theorem 1.

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