A REPRESENTATION THEOREM FOR DETERMINING FUNCTIONS

JOEL D. PINCUS AND JAMES ROVNYAK¹

Recent work has been concerned with the study of pairs of self-adjoint operators which have a one-dimensional commutator. See [1], [2] and the references cited there. This has led to a consideration of analytic functions $E(l, z)$ which are defined for $\text{Im } l \neq 0$, $\text{Im } z \neq 0$ and satisfy (A), (B), and sometimes (C).

(A) For $\text{Im } l \neq 0$, $\text{Im } z \neq 0$,

$$E(l, z)\overline{E(\bar{l}, \bar{z})} = 1.$$

(B) For $\text{Im } l \neq 0$, $\text{Im } z \neq 0$,

$$i[E(l, z)\overline{E(\bar{l}, \bar{z})} - E(l, \bar{z})\overline{E(\bar{l}, z)}]/[(l - \bar{l})(z - \bar{z})] \geq 0.$$

(C) For $\text{Im } l \neq 0$, $\text{Im } z \neq 0$,

$$|E(l, z) - 1| = O(1/|\text{Im } l \text{ Im } z|) \quad \text{as } |\text{Im } l \text{ Im } z| \rightarrow \infty.$$

In the most familiar examples, $E(l, z)$ has a known structure. In [1] we stated without proof a general structure theorem for the class of functions satisfying (A), (B), and (C). In this paper we give a proof of the theorem.

THEOREM. *Let $E(l, z)$ be an analytic function defined for $\text{Im } l \neq 0$, $\text{Im } z \neq 0$. In order that $E(l, z)$ satisfy (A) and (B) it is necessary and sufficient that*

$$(1) \quad E(l, z) = \exp \left\{ ip(l) + iq(z) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1 + xl}{x - l} \cdot \frac{1 + tz}{t - z} \frac{g(t, x) dt dx}{(1 + x^2)(1 + t^2)} \right\},$$

where $p(l)$ and $q(z)$ are analytic functions defined for $\text{Im } l \neq 0$ and $\text{Im } z \neq 0$ such that $\bar{p}(\bar{l}) = p(l)$, $\bar{q}(\bar{z}) = q(z)$, and $g(t, x)$ is a measurable

Received by the editors August 8, 1968.

¹ The present work was completed while the first author was on leave from Brookhaven National Laboratory at the Courant Institute of Mathematical Sciences of New York University, and the second author was a visiting member of the Institute for Advanced Study in Princeton, New Jersey. The research was supported by the United States Atomic Energy Commission and the National Science Foundation, and the first author also acknowledges support received from Air Force Office of Scientific Research Grant AF-AFOSR-684-64.

function of two real variables such that $0 \leq g(t, x) \leq 1$ for all real t and x . In order that $E(l, z)$ satisfy (A), (B), and (C) it is necessary and sufficient that

$$(2) \quad E(l, z) = \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g(t, x)}{(x-l)(t-z)} dt dx \right\}$$

where $g(t, x)$ is a measurable function of two real variables such that $0 \leq g(t, x) \leq 1$ for all real t and x and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(t, x) dt dx < \infty.$$

LEMMA. Let $G(l, z)$ be a real valued function which is harmonic in each variable separately such that $0 \leq G(l, z) \leq 1$ for $\text{Im } l > 0$, $\text{Im } z > 0$. Then

$$(3) \quad G(l, z) = \frac{\text{Im } l \text{ Im } z}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g(t, x)}{|x-l|^2 |t-z|^2} dx dt$$

for $\text{Im } l > 0$, $\text{Im } z > 0$, where $g(t, x)$ is a measurable real valued function of two real variables such that $0 \leq g(t, x) \leq 1$ for all real t and x .

PROOF OF LEMMA. For each $n=1, 2, 3, \dots$, we have

$$(4) \quad G(l + i/n, z + i/n) = \frac{\text{Im } l \text{ Im } z}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{G(t + i/n, x + i/n)}{|x-l|^2 |t-z|^2} dt dx,$$

$\text{Im } l > 0$, $\text{Im } z > 0$, by repeated application of the Poisson representation for a nonnegative bounded harmonic function in a half-plane. Let $g(t, x)$ be a weak*-cluster point of $(G(t + i/n, x + i/n))$, considered as a sequence in $L^\infty(R^2)$. We can always modify $g(t, x)$ on a set of measure zero so as to obtain $0 \leq g(t, x) \leq 1$ for all real t and x . We obtain (3) by passing to the limit in (4) through an appropriate subsequence.

PROOF OF THEOREM. We omit the obvious proofs that a function of the form (1) satisfies (A) and (B), and a function of the form (2) satisfies (A), (B), and (C). Conversely, let $E(l, z)$ be defined and analytic for $\text{Im } l \neq 0$, $\text{Im } z \neq 0$, and assume (A) and (B) hold. Since $E(l, z)$ never vanishes by (A),

$$E(l, z) = \exp[\phi(l, z)]$$

for some analytic function $\phi(l, z)$ which is defined for $\text{Im } l \neq 0$, $\text{Im } z \neq 0$. By (A),

$$\exp[\phi(l, z) + \bar{\phi}(\bar{l}, \bar{z})] = 1,$$

$\operatorname{Im} l \neq 0, \operatorname{Im} z \neq 0$. By (B),

$$\frac{1}{\operatorname{Im} l \operatorname{Im} z} \sin \{ \operatorname{Im} [\phi(l, z) + \bar{\phi}(l, \bar{z})] \} \geq 0,$$

$\operatorname{Im} l \neq 0, \operatorname{Im} z \neq 0$. In each of the four connected components of the domain of $E(l, z)$, $\phi(l, z)$ is determined to within an added integer multiple of $2\pi i$. By judiciously choosing $\phi(l, z)$, we may assume that

$$(5) \quad \phi(l, z) + \bar{\phi}(\bar{l}, \bar{z}) = 0, \quad \operatorname{Im} l \neq 0, \quad \operatorname{Im} z \neq 0,$$

and

$$(6) \quad 0 \leq \operatorname{Im} [\phi(l, z) + \bar{\phi}(l, \bar{z})] \leq \pi, \quad \operatorname{Im} l > 0, \quad \operatorname{Im} z > 0.$$

The function $G(l, z) = \operatorname{Im} [\phi(l, z) + \bar{\phi}(l, \bar{z})]$, $\operatorname{Im} l > 0, \operatorname{Im} z > 0$, is obviously harmonic as a function of z for each fixed l . It is also harmonic as a function of l for each fixed z , since $G(l, z) = \operatorname{Im} [\phi(l, z) - \phi(l, \bar{z})]$. By the lemma,

$$(7) \quad \operatorname{Im} [\phi(l, z) + \bar{\phi}(l, \bar{z})] = \frac{\operatorname{Im} l \operatorname{Im} z}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g(t, x)}{|x - l|^2 |t - z|^2} dt dx$$

for $\operatorname{Im} l > 0, \operatorname{Im} z > 0$, where $g(t, x)$ is a measurable function of two real variables such that $0 \leq g(t, x) \leq 1$ for all real t and x . Define

$$\phi_0(l, z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1 + xl}{x - l} \frac{1 + tz}{t - z} \frac{g(t, x) dt dx}{(1 + t^2)(1 + x^2)}$$

for $\operatorname{Im} l \neq 0, \operatorname{Im} z \neq 0$. A short calculation shows that (7) holds with $\phi(l, z)$ replaced by $\phi_0(l, z)$. Hence

$$\operatorname{Im} [\phi(l, z) + \bar{\phi}(l, \bar{z})] = \operatorname{Im} [\phi_0(l, z) + \bar{\phi}_0(l, \bar{z})]$$

for $\operatorname{Im} l > 0, \operatorname{Im} z > 0$. It follows that

$$(8) \quad \alpha(l) = \phi(l, z) + \bar{\phi}(l, \bar{z}) - \phi_0(l, z) - \bar{\phi}_0(l, \bar{z})$$

is a real valued function of l , $\operatorname{Im} l > 0$, which does not depend on z , $\operatorname{Im} z > 0$. Thus

$$\begin{aligned} \alpha(l) &= \operatorname{Re} [\phi(l, i) + \bar{\phi}(l, -i)] - \operatorname{Re} [\phi_0(l, i) + \bar{\phi}_0(l, -i)] \\ &= \operatorname{Re} [\phi(l, i) + \bar{\phi}(l, -i)] \\ &= \frac{1}{2} [\phi(l, i) + \phi(l, -i)] + \frac{1}{2} [\bar{\phi}(l, i) + \bar{\phi}(l, -i)] \\ &= \frac{1}{2} [\phi(l, i) + \phi(l, -i)] - \frac{1}{2} [\phi(\bar{l}, i) + \phi(\bar{l}, -i)] \\ &= i p(l) - i p(\bar{l}), \end{aligned}$$

where

$$p(l) = (1/2i) [\phi(l, i) + \phi(l, -i)], \quad \text{Im } l \neq 0.$$

By (5), $\bar{p}(\bar{l}) = p(l)$. By (5) we can rewrite (8) in the form

$$\phi(l, z) - \phi_0(l, z) - ip(l) = \phi(\bar{l}, z) - \phi_0(\bar{l}, z) - ip(\bar{l}),$$

$\text{Im } l > 0, \text{Im } z > 0$. By the Cauchy-Riemann equations,

$$(9) \quad iq(z) = \phi(l, z) - \phi_0(l, z) - ip(l) = \phi(\bar{l}, z) - \phi_0(\bar{l}, z) - ip(\bar{l})$$

is an analytic function of z , $\text{Im } z > 0$, which does not depend on l , $\text{Im } l > 0$. Let us define $q(z)$ for $\text{Im } z < 0$ so that $\bar{q}(\bar{z}) = q(z)$ for $\text{Im } z \neq 0$. It then follows from (5) that (9) holds for $\text{Im } l \neq 0, \text{Im } z \neq 0$. This gives the desired representation (1) for $E(l, z)$.

Now assume that $E(l, z)$ satisfies (C) also. Then there exists an integer n such that

$$\lim \phi(l, z) = 2\pi in$$

as $\text{Im } l \text{Im } z \rightarrow \infty, \text{Im } l > 0, \text{Im } z > 0$, and there exists an integer k such that

$$\lim \phi(l, z) = 2\pi ik$$

as $|\text{Im } l \text{Im } z| \rightarrow \infty, \text{Im } l > 0, \text{Im } z < 0$. Let us choose $\phi(l, z)$, as we may, so that in addition to (5) and (6) we have $n = 0$. Then $k = 0$ by (5). Now by (5),

$$\lim \phi(l, z) = 0$$

as $|\text{Im } l \text{Im } z| \rightarrow \infty, \text{Im } l \neq 0, \text{Im } z \neq 0$. We have

$$i \frac{2 \text{Im } l \text{Im } z}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g(t, x)}{|x - l|^2 |t - z|^2} dt dx = 2i \text{Im} [\phi(l, z) + \bar{\phi}(l, \bar{z})]$$

and so

$$\begin{aligned} & \left| \exp \left\{ i \frac{2 \text{Im } l \text{Im } z}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g(t, x)}{|x - l|^2 |t - z|^2} dt dx \right\} - 1 \right| \\ &= \left| \exp [\phi(l, z) + \bar{\phi}(l, \bar{z}) - \bar{\phi}(l, z) - \phi(l, \bar{z})] - 1 \right| \\ &= \left| E(l, z) \bar{E}(l, \bar{z}) E(\bar{l}, \bar{z}) \bar{E}(\bar{l}, z) - 1 \right| = O(1/|\text{Im } l \text{Im } z|) \end{aligned}$$

as $|\text{Im } l \text{Im } z| \rightarrow \infty, \text{Im } l \neq 0, \text{Im } z \neq 0$. Therefore

$$\sin \left\{ \frac{2 \text{Im } l \text{Im } z}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g(t, x)}{|x - l|^2 |t - z|^2} dt dx \right\} = O(1/|\text{Im } l \text{Im } z|)$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(\operatorname{Im} l)^2}{|x-l|^2} \frac{(\operatorname{Im} z)^2}{|t-z|^2} g(t, x) dt dx = O(1)$$

as $|\operatorname{Im} l \operatorname{Im} z| \rightarrow \infty$, $\operatorname{Im} l \neq 0$, $\operatorname{Im} z \neq 0$. By Fatou's lemma,

$$(10) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(t, x) dt dx < \infty.$$

We have already obtained

$$(11) \quad \begin{aligned} \phi(l, z) &= ip(l) + iq(z) \\ &+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1+xl}{x-l} \frac{1+tz}{t-z} \frac{g(t, x) dt dx}{(1+x^2)(1+t^2)}, \end{aligned}$$

$\operatorname{Im} l \neq 0$, $\operatorname{Im} z \neq 0$. Fix l here and let $|\operatorname{Im} z| \rightarrow \infty$. By (10),

$$\lim_{|\operatorname{Im} z| \rightarrow \infty} iq(z) = -iA$$

exists, and

$$ip(l) = iA + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1+xl}{x-l} \frac{tg(t, x) dt dx}{(1+x^2)(1+t^2)},$$

$\operatorname{Im} l \neq 0$. Substitute this last expression for $ip(l)$ in (11), fix z , and let $|\operatorname{Im} l| \rightarrow \infty$. We obtain

$$\begin{aligned} iq(z) &= -iA + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{txg(t, x) dt dx}{(1+x^2)(1+t^2)} \\ &+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1+zt}{t-z} \frac{xg(t, x) dt dx}{(1+x^2)(1+t^2)}. \end{aligned}$$

$\operatorname{Im} z \neq 0$. A short calculation now gives

$$\phi(l, z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g(t, x)}{(x-l)(t-z)} dt dx,$$

as desired.

REFERENCES

1. J. D. Pincus and J. Rovnyak, *A spectral theory for some unbounded self-adjoint singular integral operators*, Amer. J. Math. (to appear).
2. J. D. Pincus, *Commutators and systems of singular integral equations*. I, Acta Math. **121** (1968), 219-249.

COURANT INSTITUTE, NEW YORK UNIVERSITY AND
INSTITUTE FOR ADVANCED STUDY