A NOTE ON PUNCTURED DISKS IN A 2-MANIFOLD¹

EDWARD M. BROWN

Let D_n be a punctured disk with $n-1$ holes and let M be a separable 2-manifold. Let $f: D_n \rightarrow M$ be a map which is an embedding of the boundary ∂D_n of D_n .

THEOREM. *Some subfamily of the family*

$$C_1 \cup \cdots \cup C_n = f(\partial D_n)$$

of simple loops contains C_1 and bounds an embedded punctured disk in M .

This was proved in [1] for $n=1$, and in [2] for $n=1, 2$ when M is closed and orientable.

PROOF. We assume $n > 1$. Observe that $C_1 \cup \cdots \cup C_n$ is a boundary in the mod 2 homology of M . Thus any C_i is homologous to $\bigcup_{j \neq i} C_j$. If C_i were a generating loop of a handle or a Mobius band in M not meeting any other C_j , then it would generate a direct summand of the first homology group with $\bigcup_{j \neq i} C_j$ in the complementary summand. In particular each C_i is two sided.

Form a manifold N by cutting M apart along the curves C_i , $i = 2, \dots, n$ (but not C_1) and attaching disks E_i, E'_i to the resulting boundary components. Form a space $\tilde{N}$ by identifying E_i with E'_i for $i = 2, \dots, n$. Then C_1 is null homotopic in $\tilde{N}$ since $C_2, \dots, C_n$ bound disks. We show that C_1 is null homotopic in N .

Suppose not and consider what happens with the fundamental groups as we paste together one pair of disks at a time. If we bring together two components of N then we get the free product of their fundamental groups. If we paste together disks on the same component then we add a new generator and no relations. In neither case does C_1 become null homotopic.

By [1], C_1 bounds a disk D in N . Remove from D any of the E 's which it contains and take the closure of the resulting subset of M . If at most one of each pair E_i, E'_i were contained in D , then the closure would be the desired punctured disk. If some pair E_i, E'_i were contained in D , then the closure would include a handle of M having C_i as a generating loop and not meeting any other C_j . This contradiction finishes the proof.

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DARTMOUTH COLLEGE