ON EMBEDDINGS OF COMPACTA IN EUCLIDEAN SPACE

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In [5] it was shown that any two embeddings of a k -dimensional compact absolute neighborhood retract into euclidean n -space, E^n ($n \geq 5$ and $2k+2 \leq n$), are equivalent by an ambient isotopy of E^n , provided the complement of each embedding is uniformly locally 1-connected (abbreviated 1-ULC). The purpose of this paper is to prove a generalization of the result stated above for embeddings of arbitrary compacta in E^n . The following theorem is the main result.

THEOREM 1. *Suppose that X_0 and X_1 are k -dimensional compacta in E^n ($n \geq 6$ and $2k+2 \leq n$) such that $E^n - X_i$ is 1-ULC for $i=0, 1$, and $f: X_0 \rightarrow X_1$ is a homeomorphism with $d(x, f(x)) < \epsilon$ for each $x \in X_0$. Then there exists an isotopy h_t ($t \in I$) of E^n such that*

- (i) $h_0 = \text{identity}$,
- (ii) $h_t = \text{identity}$ outside the ϵ -neighborhood of X_0 for each $t \in I = [0, 1]$,
- (iii) $d(x, h_t(x)) < \epsilon$ for each $t \in I$, $x \in E^n$, and
- (iv) $h_1|_{X_0} = f$.

We shall use Gluck's terminology [8] and call the homeomorphism $h = h_1$ of the conclusion of Theorem 1 an ϵ -push of the pair (E^n, X_0) . The metric on E^n will be denoted by d . A subset Y of E^n is said to be p -ULC if for each $\epsilon > 0$ there exists $\delta > 0$ such that if ϕ is a map (i.e., continuous function) of the p -sphere S^p into Y with $\text{diam } \phi(S^p) < \delta$ then ϕ extends to a map $\bar{\phi}$ of the $(p+1)$ -ball B^{p+1} into Y with $\text{diam } \bar{\phi}(B^{p+1}) < \epsilon$. To say that Y is ULC^n means that Y is q -ULC for each $q=0, \dots, p$.

The proof of Theorem 1 parallels the proof of the main result of [5]. The only difficulty encountered is in obtaining the engulfing theorem, proved in [3], for arbitrary compacta. In this paper we incorporate the engulfing technique in the proof of Theorem 1. An engulfing theorem will appear as a corollary and not as an integral part of the proof. This method could be applied to give a more direct proof of the result in [5] for the case $n \geq 6$.

REMARK. McMillan has proved Theorem 1 in the case $k=0$ and $n \geq 5$ [12]. It is natural to conjecture that the theorem is true when $k=1$ and $n=5$. (Indeed, it seems inconceivable that the result might fail in this situation.)

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LEMMA 1. *Suppose that X is a k -dimensional compactum in E^n such that $E^n - X$ is 1-ULC. Let P be a polyhedron in E^n with $k + \dim P + 1 \leq n$ and $n - \dim P \geq 3$. Then for each $\epsilon > 0$ there exists an ϵ -push h of (E^n, X) such that $h(X) \cap P = \emptyset$.*

PROOF. (By induction on $p = \dim P$.) We know from [14] that $E^n - X$ is ULC^{n-k-2} and, hence, ULC^{p1} .

For $p = 0$, the lemma is trivial.

Given a polyhedron P of dimension p , assume inductively that X does not meet the $(p-1)$ -skeleton of P (in some arbitrarily pre-assigned triangulation of P). Let Δ be a p -simplex of P and let $\text{Int } \Delta$ and $\text{Bd } \Delta$ denote respectively the interior and boundary of Δ . Let B be a regular neighborhood of $\Delta \bmod \text{Bd } \Delta$ [7], with $B \cap P = \Delta$.

SUBLEMMA. *There exists a proper PL embedding $h: \Delta \rightarrow B$ such that $h|_{\text{Bd } \Delta} = \text{identity}$ and $h(\Delta) \cap X = \emptyset$.*

PROOF. First we construct a PL mapping $f: \Delta \rightarrow B$ such that

- (a) $f|_{\text{Bd } \Delta} = \text{identity}$,
- (b) $f(\text{Int } \Delta) \subset \text{Int } B$, and
- (c) $f(\Delta) \cap X = \emptyset$

as follows.

Since $\text{Bd } \Delta \cap X = \emptyset$, there is a polyhedral neighborhood C of $\text{Bd } \Delta$ in Δ such that $C \cap X = \emptyset$. Let $D = \text{Cl}(\Delta - C)$. Inductively, one can construct positive numbers $\epsilon_1, \epsilon_2, \dots, \epsilon_p$ and open sets $U_1, U_2, \dots, U_p$ in E^n satisfying

- (1) $D \subset U_p \subset \dots \subset U_2 \subset U_1 \subset \text{Int } B$, and
- (2) if $\phi: S^{p-i} \rightarrow U_i - X$ is a mapping with $\text{diam } \phi(S^{p-i}) < 2\epsilon_i$, then ϕ extends to a mapping $\bar{\phi}: B^{p-i+1} \rightarrow U_{i-1} - X$ such that $\text{diam } \bar{\phi}(B^{p-i+1}) < \epsilon_{i-1}$, where $U_0 = \text{Int } B$ and ϵ_0 is arbitrary.

Let T be a triangulation of D with mesh less than ϵ_p . By induction on the i -skeleton of T we obtain a mapping $f': D \rightarrow \text{Int } B - X$ such that $f'|_{\text{Bd } D} = \text{identity}$. The mapping f is then defined to be the natural extension of f' by the identity over C . Moreover, if we assume that $\text{Cl}(U_1) \subset \text{Int } B$, then the singular set of f misses $\text{Bd } \Delta$.

Next, let $g: \Delta \rightarrow B$ be a PL approximation of f such that

- (i) $g(\Delta) \subset B - X$,
- (ii) $g|_C = f|_C = \text{identity}$,
- (iii) $g(D) \subset \text{Int } B$, and
- (iv) $S_g (= \text{the singular set of } g) \text{ misses } \text{Bd } \Delta$.

Assume that g is in general position. Then $\dim S_g \leq 2p - n \leq p - 3$, and so we can find collapsible polyhedra $R \subset \text{Int } \Delta$ and $S \subset \text{Int } B$ such that

- (v) $\dim R \leq p-2$,
- (vi) $\dim S \leq p-1$,
- (vii) $S_0 \subset R$, and
- (viii) $g(R) \subset S$ [18].

Since $\dim S \leq p-1$, we may assume by our initial inductive hypothesis that $S \cap X = \emptyset$. Thus, Irwin's technique [11] can be used to construct the desired PL embedding $h: \Delta \rightarrow B - X$.

Returning now to the proof of Lemma 1 we see, by Zeeman's unknotting theorem [17], that there is an isotopy h_t ($t \in I$) of B such that $h_0 = \text{identity}$, $h_1|_{h(\Delta)} = h^{-1}$, and $h_t|_{\text{Bd } B} = \text{identity}$ for each $t \in I$. By selecting a sufficiently fine triangulation of P and applying the above procedure to each p -simplex of P that meets X , we obtain an ambient isotopy H_t ($t \in I$) of E^n with all of the desired properties.

LEMMA 2. *Suppose that X is a k -dimensional compactum in E^n ($n \geq 6$ and $2k+2 \leq n$) such that $E^n - X$ is 1-ULC, and $f: X \rightarrow E^n$ is a map with $d(x, f(x)) < \epsilon$ for each $x \in X$. Then for every $\delta > 0$ there exists an ϵ -push h of (E^n, X) such that $d(h(x), f(x)) < \delta$ for each $x \in X$.*

PROOF. This is very similar to the proof of the main lemma of [5]. For the sake of completeness we shall give the details, which are as follows.

Suppose that $f: X \rightarrow E^n$ and $\epsilon > 0$ are given as in the hypothesis of Lemma 2. Given $\delta > 0$, choose $\eta > 0$ such that if $x, y \in X$ and $d(x, y) < \eta$, then $d(f(x), f(y)) < \frac{1}{2}\delta$. Applying the Tietze extension theorem, we obtain an extension F of f to a polyhedral neighborhood N of X . By restricting F to a smaller neighborhood of X , if necessary, we can assume that $d(x, F(x)) < \epsilon$ for each $x \in N$ and that if $x, y \in N$ and $d(x, y) < \eta$, then $d(F(x), F(y)) < \frac{1}{2}\delta$. Let T be a triangulation of N with mesh less than $\frac{1}{3}\eta$. Let $q = \lfloor n/2 \rfloor - 1$, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r , and let $p = n - q - 1$. Then we have

$$2k + 2 \leq 2q + 2 \leq n, \quad k + p + 1 \leq n, \quad \text{and} \quad n - p \geq 3.$$

Let T^q and $\tilde{T}^p$ denote respectively the q -skeleton and dual $(n - q - 1)$ -skeleton of T . Then every simplex in T' , the first barycentric subdivision of T , is the join of a simplex in $(T^q)'$ with a simplex of $\tilde{T}^p$. Let $N^q = |T^q|$ (the polyhedron of T^q) and let $\tilde{N}^p = |\tilde{T}^p|$.

Assume that $F|_{N^q}$ is PL and in general position. Then $F|_{N^q}$ is a PL embedding of N^q and $d(x, F(x)) < \epsilon$ for each $x \in N^q$. Hence, by Theorem 5.5 of [1], there exists an ϵ -push h_1 of (E^n, N^q) such that $h_1|_{N^q} = F|_{N^q}$. There exists an open set U containing N^q such that if $x, y \in U$ and $d(x, y) < \eta$, then $d(h_1(x), h_1(y)) < \frac{1}{2}\delta$ (since this is true for points of N^q).

By Lemma 1, there is a $\frac{1}{3}\eta$ -push h_2 of (E^n, X) such that $h_2(X) \cap \bar{N}^p = \emptyset$ and $h_2(X) \subset \text{Int } N$. By Stallings' technique [15], there is a $\frac{1}{3}\eta$ -push h_3 of $(E^n, h_2(X))$ such that $h_3h_2(X) \subset U$.

Given $x \in X$, there exists $y \in N^a$ such that $d(x, y) < \frac{1}{3}\eta$, so that $d(h_3h_2(x), y) < \eta$. However, this means that $d(h_1h_3h_2(x), h_1(y)) < \frac{1}{2}\delta$. But $d(h_1(y), f(x)) = d(F(y), F(x)) < \frac{1}{2}\delta$, and, hence, $d(h_1h_3h_2(x), f(x)) < \delta$. If η is sufficiently small and if N is a sufficiently small neighborhood of X , then the composition $h = h_1h_3h_2$ will have all of the required properties.

The proof of Theorem 1 can now be completed by applying the technique used to prove Theorem 4.4 of [8].

As a consequence of Lemma 2, we have the following result, which is analogous to the engulfing theorem of [3].

COROLLARY. *Suppose that X is a k -dimensional compactum in E^n ($n \geq 6$, $2k+2 \leq n$) such that $E^n - X$ is 1-ULC. Suppose that $f: X \rightarrow E^n$ is a mapping with $d(x, f(x)) < \epsilon$ for each $x \in X$ and U is an open set containing $f(X)$. Then there is an ϵ -push h of (E^n, X) such that $h(U) \supset X$.*

Let M_n^k denote the set of all points in E^n at most k of whose coordinates are rational. It is proved in [10] that every k -dimensional separable metric space embeds in M_{2k+1}^k . Complementing this fact, we have the following application of Theorem 1.

THEOREM 2. *Any two embeddings of a k -dimensional compactum X into M_n^k ($n \geq 6$, $2k+2 \leq n$) are equivalent by an ambient isotopy of E^n .*

PROOF. We show that the complement of any such embedding is 1-ULC. Given $X \subset M_n^k$ as above, suppose that Γ' is a loop in $E^n - X$. Then Γ' is homotopic via an arbitrarily small homotopy (in $E^n - X$) to a polyhedral simple closed curve Γ having the property that each of its edges lies in a line whose equations are

$$x_{i_1} = r_1, \quad \dots, \quad x_{i_{n-1}} = r_{n-1},$$

where $1 \leq i_1 < \dots < i_{n-1} \leq n$ and each r_j is a rational number, and each of its vertices has all coordinates rational. Let $s = (s_1, \dots, s_n)$ be a vertex of Γ , and let f_t ($t \in I$) be the obvious deformation of E^n onto s that first deforms E^n onto the $(n-1)$ -plane $x_1 = s_1$, then onto the $(n-2)$ -plane $x_1 = s_1, x_2 = s_2$, and so on. Observe that if $p \in \Gamma$, then $f_t(p)$ has at least $n-2$ rational coordinates for each $t \in I$. Since $n-2 > k$, $f_t(\Gamma) \subset E^n - X$ for each $t \in I$. Thus $f_t|_\Gamma$ shrinks Γ to a point in $E^n - X$, and the track of this homotopy has diameter no greater than the diameter of Γ .

I am indebted to C. Lacher for pointing out the following observations relating to a paper by J. Milnor.

In [13] Milnor proves that the set of wild embeddings of the circle S^1 into E^3 contains a dense G_δ in the set of all embeddings of S^1 into E^3 , but that the opposite is true for embeddings of S^1 into E^n when $n \geq 4$. That is, most (in the Baire sense) embeddings of S^1 into E^n are tame when $n \geq 4$. More generally, he shows that most embeddings of a k -dimensional polyhedron P into E^n are tame whenever $n \geq 3k + 1$. (An embedding $f: P \rightarrow E^n$ is *tame* provided there exists a homeomorphism h of E^n such that $hf: P \rightarrow E^n$ is piecewise linear.)

As in [13] let $\text{Emb}(X, E^n)$ denote the set of all embeddings of a compactum X into E^n . Since $\text{Emb}(X, E^n)$ is the countable intersection of open subsets of the complete metric space $M(X, E^n)$ of all mappings of X into E^n , $\text{Emb}(X, E^n)$ is a Baire space (that is, it is topologically complete). In case $2\dim X + 1 \leq n$, $\text{Emb}(X, E^n)$ is in fact a dense G_δ in $M(X, E^n)$ [10]. Consequently, we can conclude from Theorem 2 and the above remarks that $\text{Emb}(X, E^n)$ ($2\dim X + 2 \leq n$) contains a dense G_δ of mutually equivalent embeddings in each of the situations:

- (a) $n \geq 6$,
- (b) $n = 5$ and $\dim X = 0$ [12],
- (c) $n \geq 5$ and X an ANR [5].

Continuing along these lines, we see from the proof of Theorem V5 of [10] that if $\dim X \leq n - 3$, then most $f \in M(X, E^n)$ map X into M_n^{n-3} (as defined above). Hence, it is clear from the proof of Theorem 2 that for most $f \in M(X, E^n)$, $E^n - f(X)$ is 1-ULC. Suppose now that X is a polyhedron. If every topological embedding of X into E^n ($\dim X \leq n - 3$) can be approximated by a PL embedding, then for most $f \in \text{Emb}(X, E^n)$, $E^n - f(X)$ is 1-ULC. Such an approximation theorem is known in case $\dim X < (2/3)n - 1$ [16] and has been announced for $\dim X \leq n - 3$, assuming that X is a closed PL manifold [9]. This together with the results of [4] and [6] proves that most embeddings of a k -dimensional polyhedron X into E^n are tame whenever $k < (2/3)n - 1$ and suggests that this phenomenon occurs up to codimension three as well. However, at the time of this writing Homma's proof of the codimension three approximation theorem [9] is still in doubt [2] so that the restriction to the meta-stable range, $k < (2/3)n - 1$, remains in effect.

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