COMPACTNESS OF LIMIT SETS AND SEMIORBIT CLOSURES

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1. Introduction. The notion of the P -limit set P_x of a point is a natural generalization of that of the omega limiting set of a point. (See [1] and [2]. See [2] for notation and terminology.) It is well known that P_x is compact and nonempty if the corresponding semi-orbit xP has compact closure. In this paper we establish (Theorem 2) the converse of this statement for transformation groups with locally compact Hausdorff phase space and generative phase group. We introduce the concept of K -connectedness and show (Theorem 1) that a replete semigroup has a translate which is K -connected. This is a generalization of a result of F. J. Hahn [3, Lemma 3.6] on replete semigroups in R^n .

2. K -connectedness. Let T be a generative topological group. Thus T is abelian and is generated by a compact neighborhood K of the identity, i.e., $T = \bigcup_{n=0}^{\infty} K^n$. We may assume without loss of generality that K is symmetric. We now fix K for the remainder of this section.

Let P be a replete semigroup in T . Define an equivalence relation $\sim$ on P as follows: $a \sim b$ provided that there exist $k_1, \dots, k_n \in K$ such that

- (1) $b = ak_1 \dots k_n$.
- (2) $ak_1 \dots k_i \in P$ for $i = 1, 2, \dots, n$.

DEFINITION. A subset Q of P is said to be K -connected provided that all the elements of Q belong to the same equivalence class.

REMARK. (1) Any subset of a K -connected set is K -connected.

(2) If A and B are K -connected and if $A \cap B \neq \emptyset$, then $A \cup B$ is K -connected.

(3) If $A_1, A_2, \dots$ are K -connected and if $A_n \cap A_{n+1} \neq \emptyset$ for $i = 1, 2, \dots$, then $\bigcup_{n=1}^{\infty} A_n$ is K -connected.

(4) If $p \in P$ and A is K -connected, then pA is K -connected.

(5) If $p \in P$ and $pK^n \subseteq P$, then pK^n is K -connected.

LEMMA 1. Let Q be a K -connected replete semigroup contained in P . Then PQ is K -connected.

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PROOF. Let $p_1, p_2 \in P$. Since Q is a replete semigroup, $T = QQ^{-1}$ [2, Theorem 6.04]. Hence $p_1 p_2^{-1} \in QQ^{-1}$ and so $p_1 Q \cap p_2 Q \neq \emptyset$. Now $p_1 Q$ and $p_2 Q$ are both K -connected, so $p_1 Q \cup p_2 Q$ is K -connected. But p_1 and p_2 were chosen arbitrarily, so $\bigcup_{p \in P} pQ = PQ$ is K -connected.

THEOREM 1. *There exists $p \in P$ such that pP is K -connected.*

PROOF. It suffices to find a K -connected replete semigroup Q contained in P . For then $pP \subseteq PQ$ for any $p \in Q$.

Choose $p \in P$ such that $pK \subseteq P$. Then $p \in K^m$ for some $m \geq 0$. Let $Q = \bigcup_{n=m}^{\infty} p^n K^n$. Then Q is a replete semigroup [2, Theorem 6.08] and it is clearly contained in P . We show that Q is K -connected.

Since $p \in K^m$ and $e \in K^{m+1}$, it follows that $p^m K^m \cap p^{m+1} K^{m+1} \neq \emptyset$, as p^{m+1} is in both sets. Now let $j \geq 0$. Then $p^{m+j} K^m \cap p^{m+j+1} K^{m+1} \neq \emptyset$, and so $p^{m+j} K^{m+j} \cap p^{m+j+1} K^{m+j+1} \neq \emptyset$. Hence by (3) and (5) of the Remark, Q is K -connected.

DEFINITION. (See [1].) A subset E of P is said to be P -extensive provided that $E \cap pP \neq \emptyset$ for all $p \in P$.

COROLLARY. *Let P_1 and P_2 be subsets of P such that no member of P_1 is in the same equivalence class as any member of P_2 . Then P_1 and P_2 cannot both be P -extensive.*

LEMMA 2. *Let P_1 and P_2 be disjoint subsets of P such that $P_1 \cup P_2 = P$. Then the following statements are equivalent:*

- (1) $P_1 K \cap P_2 = \emptyset$.
- (2) *No member of P_1 is in the same equivalence class as any member of P_2 .*

PROOF. (2) $\Rightarrow$ (1). Let $p_1 \in P_1, k \in K$. If $p_1 k \in P$, then $p_1 k \sim p_1$, hence $p_1 k \notin P_2$. If $p_1 k \notin P$, then $p_1 k \notin P_2$.

(1) $\Rightarrow$ (2). Let $p_1 \in P_1, p_2 \in P_2$ and $p_1 \sim p_2$. Thus there exists $k_1, \dots, k_n \in K$ such that $p_1 k_1 \dots k_n = p_2$ and $p_1 k_1 \dots k_i \in P$ for $i = 1, 2, \dots, n$. Hence $p_1 k_1 \in P_1 K \cap P = P_1$, and so $p_1 k_1 k_2 = (p_1 k_1) k_2 \in P_1$. Similarly $p_1 k_1 \dots k_i \in P_1$ for $i = 1, 2, \dots, n$. But this contradicts $p_2 \in P_2$.

3. Theorem on compactness of limit sets and semiorbit closures.

DEFINITION. (See [2].) Let $x \in X$ and let P be a replete semigroup in T . Then the P -limit set of x is defined by

$$P_x = \bigcap_{t \in T} \text{cl}(xtP).$$

It is easy to show that P_x is closed and invariant, and that $P_x = \bigcap_{p \in P} \text{cl}(xpP)$.

THEOREM 2. *Let (X, T, π) be a transformation group where X is a locally compact Hausdorff space and T is generative. Let $x \in X$ and let P be a replete semigroup in T . Then P_x is compact and nonempty if and only if $\text{cl}(xP)$ is compact.*

PROOF. It suffices to show that if P_x is compact and nonempty, then $\text{cl}(xP)$ is compact. So, let U be an open neighborhood of P_x which has compact closure, and let K be a compact symmetric neighborhood of the identity which generates T . Let $A = \overline{U}K - U$, and so A is compact. For each $p \in P$, let $A(p) = A \cap \text{cl}(xpP)$. Thus each $A(p)$ is a compact subset of A . Now

$$\bigcap_{p \in P} A(p) = A \cap \bigcap_{p \in P} \text{cl}(xpP) = A \cap P_x = \emptyset$$

so by the finite intersection property, there exist $p_1, p_2, \dots, p_n \in P$ such that $A(p_1) \cap \dots \cap A(p_n) = \emptyset$. Let $p_0 = p_1 p_2 \dots p_n$ and thus $A(p_0) = \emptyset$. Hence $x p_0 P \subseteq X - A = U \cup (X - \overline{U}K)$. Let

$$P_1 = \{p \in P \mid x p_0 p \in U\} \quad P_2 = \{p \in P \mid x p_0 p \in X - \overline{U}K\}.$$

Then $P_1 \cup P_2 = P$ and $P_1 \cap P_2 = \emptyset$.

Let $p_1 \in P_1$ and $k \in K$. Then $x p_0 p_1 k \in UK$ and so $p_1 k \notin P_2$. Thus $P_1 K \cap P_2 = \emptyset$ and then by Lemma 2, no member of P_1 is in the same equivalence class as any member of P_2 . Hence by the corollary, P_1 and P_2 cannot both be P -extensive.

We now show that P_1 is P -extensive. Since P_x is invariant, $U p_0^{-1}$ is a neighborhood of P_x . As $P_x \neq \emptyset$, $\{p \in P \mid x p \in U p_0^{-1}\}$ is P -extensive. However this last set is precisely P_1 . Thus we conclude that P_2 is not P -extensive, i.e., that there exists $p \in P$ such that $P_2 \cap pP = \emptyset$. Hence $pP \subseteq P_1$. Then $x p P \subseteq x P_1 \subseteq U p_0^{-1}$. Thus $xP \subseteq U p_0^{-1} p^{-1}$ and so $\text{cl}(xP)$ is compact.

REFERENCES

1. John D. Baum, *P-recurrence in topological dynamics*, Proc. Amer. Math. Soc. **7** (1956), 1146-1154.
2. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloq. Publ. no. 36, Amer. Math. Soc., Providence, R. I., 1956.
3. Frank J. Hahn, *Some embeddings, recurrence properties, and the Birkhoff-Markov Theorem*, Duke Math. J. **27** (1960), 513-525.

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