AN INFINITE DIMENSIONAL VERSION OF A THEOREM OF BERNSTEIN

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1. Introduction. Let $\mathcal{P}(R^n)$ be the algebra of polynomials in n variables with the topology of uniform convergence on bounded sets of a function f and its derivative f' . A classical theorem of Bernstein says that the closure of $\mathcal{P}(R^n)$ is the algebra $\mathcal{C}^1(R^n)$ of all real-valued functions of class C^1 . In other words, for every $f \in \mathcal{C}^1(R^n)$ there is a sequence $\{p_n\}$ of polynomials such that $p_n \rightarrow f$ uniformly on bounded sets and $f'_n \rightarrow f'$ uniformly on bounded sets. In this paper we define the algebra $\mathcal{P}(X)$ of polynomials in a Banach space X and determine its closure for a restricted class of reflexive Banach spaces (Theorem 8). Thus, Theorem 8 answers a question raised by the author in [2]. In what follows, weak convergence in X will be denoted by $x_n \rightharpoonup x$ and strong convergence by $x_n \rightarrow x$. Henceforth we will assume all Banach spaces to be separable.

2. Polynomials in Banach spaces. Let X be a Banach space and let X^* be its dual. If $u = (u_1, \dots, u_n) \in (X^*)^n$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ is a sequence of positive integers, we define u^α to be the real valued function $u^\alpha(x) = (u_1(x))^{\alpha_1} \dots (u_n(x))^{\alpha_n}$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$. A polynomial in X is a function $p: X \rightarrow R$ of the form $p = \sum_{|\alpha| \leq m} a_\alpha u^\alpha$ where a_α is a real number.

Let X, Y be Banach spaces. An operator $T: X \rightarrow Y$ is *weakly continuous* if T is continuous from the weak topology in X to the norm topology in Y .

THEOREM 1. *Let X be a Banach space and let p be a polynomial. Then*

- (i) *p is weakly continuous.*
- (ii) *The derivative $p': X \rightarrow X^*$ is weakly continuous.*
- (iii) *$p'(X)$ is finite dimensional.*

PROOF. (i) follows from the definition of p . In order to prove (ii) and (iii) we simply calculate $p'(x)$. If $p(x) = u(x)v(x)$, where $u, v \in X^*$, then $p'(x) = u(x)v + v(x)u$, and if $p(x) = (u(x))^k$ then $p'(x) = k(u(x))^{k-1}u$. Now, by induction, one can show if $p = u^\alpha = u_1^{\alpha_1} \dots u_n^{\alpha_n}$ then $p'(x)$ is of the form $\sum_{i=1}^n \lambda_i(x)u_i$ where λ_i is a weakly continuous real valued function. In this case p' satisfies (ii) and (iii). The general case follows from the previous ones.

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THEOREM 2. *Let X, Y be Banach spaces and $f: X \rightarrow Y$ continuous. Let $P: X \rightarrow Y$ be linear and bounded and such that $P(X)$ is finite dimensional. Then*

(i) *$f \circ P$ is weakly continuous.*

(ii) *$(f \circ P)' = P^* \circ f' \circ P$.*

PROOF. (i) Let $x_n \rightarrow x$ (x_n converges weakly to x). Then $Px_n \rightarrow Px$ since P is linear. Furthermore, $Px_n \rightarrow Px$ since $P(X)$ is finite dimensional, so $fP(x_n) \rightarrow fP(x)$ since f is continuous.

(ii) A trivial calculation.

THEOREM 3. *Let X be a reflexive Banach space. Then $f: X \rightarrow \mathbb{R}$ is weakly continuous on bounded sets if and only there is a sequence $\{p_n\}$ of polynomials such $p_n \rightarrow f$ uniformly on bounded sets.*

PROOF. Every set of the form $\{x \mid \|x\| \leq M\}$ is weakly compact, so f is weakly continuous if $p_n \rightarrow f$ uniformly on bounded sets. Let us prove the second part. The theorem of Stone-Weierstrass can be used to show that for every continuous function f on bounded sets and every integer n there is a polynomial p_n such that $|f(x) - p_n(x)| < 1/n$ for all x such that $\|x\| \leq n$. The sequence $\{p_n\}$ converges to f uniformly on bounded sets. This concludes the proof.

REMARK. Weakly continuous on bounded sets means: the restriction of f to any bounded subset of X is relatively weakly continuous. If a function is weakly continuous, then it is weakly continuous on bounded sets. The converse is not true. Let $X = H$ be a Hilbert space, let $e_1, e_2, \dots, e_n, \dots$ be a basis of H and let

$$f(x) = \sum_{n=1}^{\infty} \frac{(e_n(x))^n}{(2n)!}.$$

If this function were weakly continuous at the origin of H then it would be bounded on some subspace M of finite codimension in H . For $x \in M$ fixed,

$$g(z) = \sum_{n=1}^{\infty} \frac{(e_n(x))^n}{(2n)!} z^n$$

is an entire function of order $1/2$ in z . Such a function cannot be bounded for all real z unless it is a constant, in this case zero. Hence $M = \{0\}$ which is a contradiction. I am grateful to the referee for pointing out this nice example and for other constructive comments.

We say that a Banach space X has *property (B)* if there is a sequence $P_n: X \rightarrow X$ of bounded linear operators such that

- (i) $P_n(X)$ is finite dimensional for each n .
- (ii) $P_n^2 = P_n$.
- (iii) $P_n(x) \rightarrow x$ for every $x \in X$.
- (iv) $P_n^*(u) \rightarrow u$ for every $u \in X^*$.

Every Banach space with a biorthogonal basis has *property (B)*. In particular, every Hilbert space has *property (B)*. The theorem below has been proved by Vainberg in [1] for reflexive Banach spaces with a biorthogonal basis. Our proof is similar to his.

THEOREM 4. *Let X be a reflexive Banach space with property (B). Let $T: X \rightarrow X^*$ be a continuous operator. Then T is weakly continuous on bounded sets if and only if $T \circ P_n \rightarrow T$ uniformly on bounded sets.*

PROOF. By the uniform boundness principle there is a constant M such that $\|P_n\| \leq M$ and $\|P_n^*\| \leq M$ for all n . Let us show that if $x_n \rightarrow x$ then $P_n x_n \rightarrow x$. Now, $\langle P_n x_n, u \rangle = \langle x_n, P_n^* u \rangle = \langle x_n, u \rangle + \langle x_n, P_n^* u - u \rangle$. Since $P_n^* u \rightarrow u$, and $\{x_n\}$ is bounded, it follows that $\langle P_n x_n, u \rangle \rightarrow 0$ for every $u \in X^*$.

Let us assume that T is weakly continuous on bounded sets and suppose that $T \circ P_n$ does not converge to T uniformly on bounded sets. Then, for some $\epsilon > 0$ and for every integer k there exist an integer n_k and a point x_k such that $\|x_k\| \leq M$ for some $M > 0$ and $\|TP_{n_k} x_k - Tx_k\| \geq \epsilon$, $k = 1, 2, \dots$. Since X is reflexive, there is a subsequence (still denoted by $\{x_k\}$) such that $x_k \rightarrow x_0$ for some x_0 . This would imply that $P_{n_k} x_k \rightarrow x_0$ and $TP_{n_k} x_k \rightarrow Tx_0$. Thus, we have reached a contradiction. Let us assume that $T \circ P_n \rightarrow T$ uniformly on bounded sets. Since $T \circ P_n$ is weakly continuous by Theorem 2 and $\{x \mid \|x\| \leq M\}$ is weakly compact for every $M > 0$, it follows that T is weakly continuous on bounded sets.

3. Uniformly differentiable functions. Let X and Y be Banach spaces and let $f: X \rightarrow Y$ be of class C^1 . We say that f is *uniformly differentiable* in a subset $A \subset X$ if for every $\epsilon > 0$ there is some $\delta > 0$ such that

$$\|f(a+h) - f(a) - \langle f'(a), h \rangle\| \leq \epsilon \|h\|$$

whenever $\|h\| \leq \delta$ for all $a \in A$.

THEOREM 5. *Let X be a reflexive Banach space with property (B). Let $f: X \rightarrow R$ be weakly continuous and uniformly differentiable on bounded sets. Let $T = f': X \rightarrow X^*$. Then $P_n^* \circ T \rightarrow T$ uniformly on bounded sets.*

PROOF. Let $I: X \rightarrow X$ be the identity operator and let $h = (I - P_n)(z)$.

Let $O(x, h) = f(x+h) - f(x) - \langle T(x), h \rangle$. Since $\langle Tx, h \rangle = \langle (I - P_n^*)Tx, h \rangle$ we get,

$$(1) \quad \begin{aligned} |\langle (I - P_n^*)Tx, h \rangle| &\leq |f(x+h) - f(x)| + |O(x, h)| \\ &\leq |f(x+h) - f(P_n(x+h))| \\ &\quad + |f(P_n(x+h) - f(x)| + |O(x, h)|. \end{aligned}$$

Let $\epsilon > 0$ be given and let $M > 0$. Since f is uniformly differentiable on bounded sets there is some $\delta > 0$ such that

$$(2) \quad |O(x, h)| < (\epsilon/3)\|h\| \quad \text{if } \|h\| < \delta$$

for all x such that $\|x\| \leq M$. Now, by the uniform boundness principle there is some N such that $\|I - P_n\| \leq N$ for all n . Since f is weakly continuous, there is some n_0 (see Theorem 4) such that

$$(3) \quad \begin{aligned} |f(x+h) - f(P_n(x+h))| &< \epsilon\delta/3N \\ |f(x) - f(P_n x)| &< \epsilon\delta/3N \end{aligned} \quad n \geq n_0$$

for all x such that $\|x\| \leq N$ and all h such that $\|h\| < \delta$. Let $y \in X$ such that $\|y\| = 1$ and let $h = (\delta/N)(I - P_n)y$. then $\|h\| \leq \delta$ and

$$(4) \quad \begin{aligned} |\langle (I - P_n^*)Tx, y \rangle| &= |\langle (I - P_n^*)Tx, (I - P_n)y \rangle| \\ &= (M/\delta) |\langle (I - P_n^*)Tx, h \rangle|; \end{aligned}$$

from (1), (2), (3) and (4) we get

$$\|(I - P_n^*)Tx\| < \epsilon \quad n \geq n_0$$

for all x such that $\|x\| \leq M$. This concludes the proof.

THEOREM 6. *Let X be a reflexive Banach. Assume $f: X \rightarrow R$ is weakly continuous and uniformly differentiable on bounded sets. Then $f' = T: X \rightarrow X^*$ is weakly continuous.*

PROOF. We have to show that $x_n \rightarrow x$ implies $Tx_n \rightarrow Tx$. Since $\{x_n\}$ is bounded, we can assume that everything takes place in some bounded set $B = \{x \mid \|x\| \leq M\}$. Let $\epsilon > 0$. By uniform differentiability we can make

$$(1) \quad \|O(x, h)\| < 4^{-1}\epsilon\|h\|$$

if $\|h\| \leq \delta < 1$ for some δ and all $x \in B$. Since f is weakly uniformly continuous in $2B$, there is a weak neighborhood $V(0)$ such that,

$$(2) \quad |f(y) - f(z)| < 4^{-1}\delta\epsilon$$

for all $y, z \in 2B$ such that $y - z \in V(0)$. Now,

$$\langle Tx_n, h \rangle = f(x_n + h) - f(x_n) - O(x_n, h)$$

$$\langle Tx, h \rangle = f(x + h) - f(x) - O(x, h)$$

and

$$(3) \quad \begin{aligned} |\langle Tx_n - Tx, h \rangle| &\leq |f(x_n + h) - f(x + h)| + |f(x_n) - f(x)| \\ &\quad + |O(x_n, h)| + |O(x, h)|. \end{aligned}$$

Let n_0 be such that $x_n - x_0 \in V$ if $n \geq n_0$. From (1), (2) and (3) we obtain

$$|\langle Tx_n - Tx, h \rangle| < \epsilon \delta$$

if $n \geq n_0$ and $\|h\| = \delta$. This shows that $|\langle Tx_n - Tx, h \rangle| < \epsilon$ if $n \geq n_0$ and $\|h\| = 1$. Therefore, $\|Tx_n - Tx\| < \epsilon$ if $n \geq n_0$. This concludes the proof.

THEOREM 7. *Let X be a reflexive Banach with property (B). Let $f: X \rightarrow R$ be weakly continuous and uniformly differentiable on bounded sets. Then $P_n^*TP_n \rightarrow T$ uniformly on bounded sets.*

PROOF. We have for any $x \in X$

$$\|Tx - P_n^*TP_n(x)\| \leq \|Tx - P_n^*Tx\| + \|P_n^*Tx - P_n^*TP_nx\|.$$

The first term in the right-hand side converges to zero uniformly on bounded sets (Theorem 5). The second term is lesser or equal than $\|P_n^*\| \|Tx - TP_nx\|$ which converges to zero uniformly on bounded sets by Theorems 6 and 4 and the fact that $\|P_n\| \leq M$ for some constant M and all n . This concludes the proof.

4. The main theorem.

THEOREM 8. *Let X be a reflexive Banach space with property (B). Let $\mathcal{P}(X)$ be the algebra of polynomials in X with the topology of uniform convergence on bounded sets of a function and its derivative. Then the closure of (X) is the algebra of weakly continuous functions on bounded sets which are uniformly differentiable on bounded sets.*

PROOF. Let f be weakly continuous and uniformly differentiable on bounded sets, and let $f_n = f \circ P_n$. Then $f'_n = P_n^* \circ f' \circ P_n$ (Theorem 2), $f_n \rightarrow f$ uniformly on bounded sets (Theorem 4) and $f'_n \rightarrow f'$ uniformly on bounded sets (Theorem 7). Therefore, we can assume that that $f = g \circ P$ where P is a projection into some finite-dimensional subspace and g is uniformly differentiable on bounded sets. Let $P(X) = H$ and let h be the restriction of g to H , that is, $h(x) = g(x)$ for all $x \in H$. Since H is finite dimensional there is a sequence $\{q_n\}$ of

polynomials in H such that $q_n \rightarrow h$, $q'_n \rightarrow h'$ uniformly on bounded sets (this follows from a classical theorem of Bernstein). Let $p_n = q_n \circ P$. Then,

$$|f(x) - p_n(x)| = |gP(x) - q_nP(x)| \rightarrow 0$$

uniformly on bounded sets. Now, let $u \in X$ such that $\|u\| = 1$. Then,

$$\begin{aligned} |\langle f'(x) - p'_n(x), u \rangle| &= |\langle g'Px - q'_nPx, Pu \rangle| \\ &= |\langle h'P(x) - q'_nPx, Pu \rangle| \\ &\leq \|P\| \|h'P(x) - q'_nPx\|. \end{aligned}$$

Therefore, $\|f', (x) - p'_n(x)\| \leq \|P\| \|h'P(x) - q'_nPx\| \rightarrow 0$ uniformly on bounded sets. This concludes the proof.

REFERENCES

1. M. M. Vainberg, *Variational methods for the study of nonlinear operators*, Holden-Day, San Francisco, Calif., 1964.
2. G. Restrepo, *Remarks on weakly continuous functions in Banach spaces*, Rev. Colombiana Mat. (to appear).

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