THE BREADTH AND DIMENSION OF A TOPOLOGICAL LATTICE

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E. Dyer and A. Shields [7] conjectured that if L is a compact connected metrizable distributive topological lattice, then $\dim(L)$ is equal to the breadth of L . L. W. Anderson [1] has proved that if L is a locally compact, (chain-wise) connected distributive topological lattice then the breadth of L is less than or equal to the codimension of L .

In this note we shall show that if L is a locally compact connected distributive topological lattice of inductive dimension n and if the set of points at which L has dimension n has nonvoid interior, then the breadth of L is also n .

Our terminology and notation used in this paper are the same as in [1] and [4].

It is well known that the number of elements in a finite Boolean algebra is always a power of two, and if there are 2^n elements then there are exactly n atoms.

It is also known that the center C of any lattice with 0 and 1 forms a Boolean lattice with the same 0 and 1, and that if L is a compact connected topological lattice of codimension n then the cardinal of C of L , hereafter denoted by $\text{Card}(\text{Cen}(L))$, is at most 2^n , [4].

The following two lemmas have appeared in [4].

LEMMA 1. *Let L be a compact connected distributive topological lattice. Then L is isomorphic (homeomorphic and lattice-isomorphic) with a Cartesian product of n nondegenerate compact connected chains (hereafter called an n -cube) iff (i) the center of L has exactly n atoms and (ii) for each atom x of the center $x \wedge L$ is a chain.*

LEMMA 2. *Let L be a distributive topological lattice with 0 and 1. If $\{x_1, \dots, x_n\}$ is the set of all the atoms of the center of L and if x and y are two incomparable elements in $x_i \wedge L$ for some i , then $\text{Card}(\text{Cen}(L)) < \text{Card}(\text{Cen}([x \wedge y, x \vee y \vee c(x_i)]))$, where $c(x_i)$ is the complemented element of x_i in the center of L .*

Hereafter, inductive dimension is referred to as dimension. In a space L of dimension n , L_n denotes the set of all points at which L has dimension n . If L is a topological lattice and $[x, y] \subset L$, and if $\text{Card}(\text{Cen}([x, y])) = 2^n$, then $\text{Ca}([x, y])$ will denote the cardinal n of the atoms of $\text{Cen}([x, y])$. It is known [6] that the codimension is

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less than or equal to the inductive dimension in a locally compact Hausdorff space.

THEOREM. *If L is a locally compact connected distributive topological lattice of dimension n and if L_n has a nonvoid interior, then L contains a subset which is isomorphic with an n -cube.*

PROOF. Let W_1 be a nonvoid open subset of L and $W_1 \subset L_n$. For an element $w \in W_1$ choose neighborhoods V_1 and U_1 of w and a closed interval $[a, b]$ such that V_1 is convex, U_1^* compact and $V_1 \subset [a, b] \subset U_1^* \subset W_1$, (see [5]), where U_1^* denotes the closure of U_1 . Then $M = [a, b]$ is a compact connected distributive topological lattice of dimension n in its relative topology. Consider $N = \{m \mid m = \text{Ca}([x, y]) \text{ for a closed interval } [x, y] \subset V_1\}$. Then N is bounded by n . Let m be the greatest integer of N , $m = \text{Ca}([\alpha, \beta])$, and let $\{x_1, \dots, x_m\}$ be the set of atoms of the center of $[\alpha, \beta]$. Now we shall show that for each i , $I_i = x_i \wedge [\alpha, \beta]$ is a nondegenerate compact connected chain. In fact, $[\alpha, \beta]$ is a distributive sublattice. Suppose that I_i is not a chain for some i . Then, by Lemma 2, there exists a closed interval $[\alpha', \beta']$ in $[\alpha, \beta]$ such that $m < \text{Ca}([\alpha', \beta'])$. This is a contradiction. For each i , I_i is compact and connected because $[\alpha, \beta]$ is. Moreover, the relative topology of each I_i is its intrinsic topology, (see [3]). Since M is compact, $[\alpha, \beta]$ is a compact connected distributive topological lattice in its relative topology. By Lemma 1, it follows that $[\alpha, \beta]$ is isomorphic with an m -cube $I^m = I_1 \times I_2 \times \dots \times I_m$ under a mapping f .

Suppose $m < n$. Let p be an element of $[\alpha, \beta]$ such that the i th coordinate of $f(p)$ is an inner point of I_i , $i = 1, 2, \dots, m$, (such a point p is called an inner point of I^m). Now we choose a convex neighborhood U_2 of p such that

$$U_2 \subset L \setminus \{ (c \wedge L) \cup (c \vee L) \mid c \in \text{Cen}([\alpha, \beta]) \text{ and } c \neq \alpha, c \neq \beta \}.$$

Setting $U = U_2 \cap V_1$, we choose a neighborhood W of p such that $W \vee W \subset U$ and $W \wedge W \subset U$. If we set $V = U \cap [\alpha, \beta]$, then V is a convex neighborhood of p in $[\alpha, \beta]$, and the dimension of V is at most m . Thus $W \setminus V \neq \emptyset$. For an element $z \in W \setminus V$ we have either $z \vee p \notin V$ or $z \wedge p \notin V$ because, otherwise, $z \in V$. Let us first assume that $z \vee p \notin V$. But $z \vee p \in U$. Set $\gamma = (z \vee p) \wedge \beta$. Then $p \leq \gamma < z \vee p$. Since U is convex and contains p and $z \vee p$, we have $\gamma \in V = U \cap [\alpha, \beta]$. Clearly γ is an inner point of I^m . So $[\gamma, \beta]$ is also isomorphic with a sub m -cube J^m of I^m . Consider a compact connected chain $C(\gamma, \delta)$ from γ to $\delta = z \vee p$. Then $C(\gamma, \delta)$ is nondegenerate. Moreover, we have $\delta \wedge [\gamma, \beta] = \{\gamma\}$. Therefore, the mapping $g: Q^{m+1} = [\gamma, \beta] \times C(\gamma, \delta)$

$\rightarrow P^{m+1} = [\gamma, \beta] \vee C(\gamma, \delta)$ defined by $g(x, s) = x \vee s$ is an isomorphism, where $g^{-1}(y) = (y \wedge \beta, y \wedge \delta)$. Since $g^{-1}(U \cap P^{m+1})$ is nonvoid and open in Q^{m+1} , and Q^{m+1} is isomorphic with an $(m+1)$ -cube $J^m \times C(\gamma, \delta)$, $U \cap P^{m+1}$ contains a closed interval which is isomorphic with an $(m+1)$ -cube, and which is contained in V_1 . Moreover, $\text{Ca}(J^m \times C(\gamma, \delta)) = m+1$. This is a contradiction to the fact of the maximality of m . For the case that $z \wedge p \notin V$, the dual argument also leads to a contradiction. Hence we have $m = n$ which completes the proof.

COROLLARY 1. *Under the same hypothesis of the Theorem, the breadth, the dimension and the codimension are all the same.*

It is known [8] that if L is a compact separable space of dimension n , then L_n has dimension n , and that any n -dimensional subset of Euclidean n -space has nonvoid interior.

Hence we have the following corollary:

COROLLARY 2. *If L is a locally compact connected distributive topological lattice of dimension n which is topologically contained in Euclidean n -space, then L contains a subset which is isomorphic with an n -cell.*

REMARK. It is a natural question that if L is a compact connected distributive topological lattice of dimension n , then L_n has nonvoid interior. This question is still open. Professor J. T. Borrego has, however, given an example of a compact connected, locally connected, acyclic topological semilattice of 2-dimensions whose L_2 has void interior.

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