A REMARK ON NEUWIRTH AND NEWMAN'S PAPER: "POSITIVE $H^{1/2}$ FUNCTIONS ARE CONSTANTS"

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1. In this note we prove

THEOREM 1. *Let $f(z)$ be of Hardy class $H^{1/2}$ in $U: |z| < 1$. Let I be an open arc of $K: |z| = 1$ and let the boundary values of $f(z)$ be nonnegative a.e. on I . Then $f(z)$ has an analytic continuation into D , the complement of $K - I$ with respect to the extended z -plane.*

Combined with Liouville's theorem this gives the following result due to Neuwirth and Newman [1]:

THEOREM 2. *If $f(z)$ is of Hardy class $H^{1/2}$ in U and if the boundary values of $f(z)$ are nonnegative a.e. on K , then f is a constant.*

2. **LEMMA.** *If $f_1(z)$ and $f_2(z)$ are of Hardy class H^1 in U and if their boundary values satisfy*

$$f_1(e^{i\theta}) = f_2(e^{i\theta})$$

a.e. on I , then both f_1 and f_2 can be continued analytically into D .

PROOF. By a theorem of Rudin a function $g \in H^1$ in U whose boundary values are real a.e. on I can be analytically continued to D [3, p. 59]. The lemma follows on applying Rudin's result to $g_1 = (1/2)(f_1 + f_2)$ and $g_2 = (i/2)(f_1 - f_2)$.

PROOF OF THEOREM 1. By a well-known decomposition theorem [2, p. 87], $f(z) = B(z)F^2(z)$, where $B(z)$ is a Blaschke product and $F(z) \in H^1$. Since the boundary values of $B(z)$ have absolute value one a.e. on K , we have a.e. on I , $f(e^{i\theta}) = |f(e^{i\theta})|$, or $B(e^{i\theta})F^2(e^{i\theta}) = |F^2(e^{i\theta})|$, and hence

$$B(e^{i\theta})F(e^{i\theta}) = \overline{F(e^{i\theta})}.$$

Since both BF and F are in H^1 , it follows from the lemma that BF and F can be continued to D and the same is true of $f = BF \cdot F$.

REFERENCES

1. J. Neuwirth and D. J. Newman, *Positive $H^{1/2}$ functions are constants*, Proc. Amer. Math. Soc. **18** (1967), 958.
2. F. Riesz, *Über die Randwerte einer analytischen Funktion*, Math. Z. **18** (1923), 87-95.
3. W. Rudin, *Analytic functions of class H_p* , Trans. Amer. Math. Soc. **78** (1955), 46-66.

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