A NOTE ON FREE GROUPS

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The object of this note is to point out a theorem of M. Hall, Jr. (Theorem 1), proved, but formulated in a weaker form, as Theorem 5.1 of [2]. We then show that results of Karrass and Solitar [5], and Howson [4], follow as relatively easy corollaries of this stronger statement (Corollaries 2 and 3).

Since the terminology is not fixed, we note for definiteness that by a *right transversal* for a subgroup H in a group G we shall mean a complete set of representatives of cosets Hg, $g \in G$. Other terms used are defined in [1] and [6].

The strengthened theorem is as follows.

THEOREM 1. *Let F be any free group, H any finitely generated subgroup and $a_1, \cdots, a_k$ any finite number k of elements of F not in H. Then there exists a subgroup G of finite index in F such that G avoids the set $\{a_1, \cdots, a_k\}$ and contains H as a free factor.*

We sketch Hall's proof, modified slightly, with his reference to results of [3] replaced by a reference to the following converse of Schreier's theorem on subgroups of free groups.

LEMMA 1. *Let F be a free group on a set S of free generators and let T be any right Schreier system in terms of S (i.e. a set closed under taking left initial segments, including the identity 1). If there exists a function $\phi: TS \to T$ such that*

$$\phi(ts) = ts \quad \text{if } ts \in T \quad (t \in T,\ s \in S), \tag{1}$$

and the mapping $\pi(s): t \to \phi(ts)$ is a permutation of T for each $s \in S$, then those elements $ts\,\phi(ts)^{-1}$ ($t \in T$, $s \in S$) which are nontrivial, are distinct and form a set of free generators of a subgroup which has T as a right Schreier transversal.

This is merely a combination of Theorems 7.2.3 and 7.2.6 of [1].

PROOF OF THE THEOREM. Let T_1 be a right Schreier transversal for H in F and $\phi_1: F \to T_1$ the function associating with each element of F its representative in T_1. By Schreier's theorem, H is freely generated by those elements

$$t_1 s \phi_1(t_1 s)^{-1} \qquad (t_1 \in T_1,\ s \in S) \tag{2}$$

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which are nontrivial. Let $T_2 \subseteq T_1$ be the set of all those elements t_1, $\phi_1(t_1 s)$ of T_1 such that $t_1 s \phi_1(t_1 s)^{-1} \neq 1$, together with 1 and the representatives in T_1 of the cosets $Ha_1, \cdots, Ha_k$. Since H is finitely generated and k is finite, it follows that T_2 is finite. Let T be the result of adjoining to T_2 all (left) initial segments of elements of T_2. Clearly T is finite.

We now define a function $\phi: TS \to T$. Whenever $\phi_1(ts) \in T$ ($t \in T$, $s \in S$), set $\phi(ts) = \phi_1(ts)$. This ensures that ϕ satisfies (1). Also, since the mapping $\pi_1(s): t \to \phi_1(ts) (t \in T)$ is certainly (1, 1), it follows that the partial function $\pi'(s)$ from T to T which maps t to $\phi_1(ts)$ whenever $\phi_1(ts) \in T$, can be extended to a permutation of T. For each s take an arbitrary permutation $\pi(s): T \to T$, containing $\pi'(s)$. It is easy to check that the equation $\phi(ts) = t\pi(s)$ then defines ϕ uniquely at every element of TS.

The mapping ϕ has been constructed to satisfy the conditions of Lemma 1. Therefore the elements

$$1 \neq ts\phi(ts)^{-1} \qquad (t \in T, s \in S) \tag{3}$$

freely generate a subgroup G with right Schreier transversal T (and therefore G has finite index). Among the expressions (3) occur all the free generators (2) of H. Hence H is a free factor of G.

Finally we show that $a_i \notin G$ for $i = 1, \cdots, k$. For each a_i there is a nontrivial $t_i \in T$ such that $a_i t_i^{-1} \in H$, and hence *a fortiori* such that $a_i t_i^{-1} \in G$. If $a_i \in G$ then $t_i \in G$. But t_i is a nontrivial member of a Schreier transversal for G in F and therefore $t_i \notin G$. Thus $a_i \notin G$ and the proof is complete.

We have the following corollaries.

COROLLARY 1. *If H is a finitely generated subgroup of a free group F, then H is a free factor of some subgroup of finite index in F.*

COROLLARY 2 (KARRASS AND SOLITAR [5]). *If in a free group F, H is a subgroup of infinite index containing a nontrivial normal subgroup of F, then H has infinite rank.*

PROOF. Denote by N the nontrivial normal subgroup of F, contained in H. Suppose H has finite rank. Then by Corollary 1 it is a free factor of a subgroup G of finite index. Since H has infinite index, H is a proper free factor of G. Thus N is normal in G and contained in a proper free factor of G. This is clearly impossible.

COROLLARY 3 (HOWSON [4]). *If A and B are finitely generated subgroups of a free group F, then $A \cap B$ is finitely generated.*

For the proof we need the following lemma (see [**6**, p. 117, Exercise 32]).

LEMMA 2. *If K_1 is a free factor of a free group K and G is any subgroup of K, then $K_1 \cap G$ is a free factor of G.*

PROOF OF COROLLARY 3. We may assume F has finite rank (if not replace F by sgp $\{A, B\}$). By Corollary 1 there exist subgroups $A * A_1$ and $B * B_1$ having finite indices in F (and therefore also finite ranks). Applying Lemma 2 with $A * A_1$, A and $(A * A_1) \cap (B * B_1)$ replacing K, K_1 and G respectively, we deduce that $A \cap (A * A_1) \cap (B * B_1), = A \cap (B * B_1)$, is a free factor of $(A * A_1) \cap (B * B_1)$. A second application, with $B * B_1$, B and $A \cap (B * B_1)$ replacing K, K_1 and G respectively, yields that $A \cap B$ is a free factor of $A \cap B * B_1$ and hence of $(A * A_1) \cap (B * B_1)$. But the latter group is of finite rank (since it has finite index in F) and hence so is $A \cap B$. This completes the proof.

REMARKS. 1. Suppose in the above that rank $A = m$ and rank $B = n$. Note that in the preceding proof $A \cap (B * B_1)$ has finite index (i, say) in A. Therefore by Schreier's formula,

$$\text{rank } A \cap (B * B_1) = i(m - 1) + 1.$$

Thus this certainly also provides a bound for rank $A \cap B$. However, Howson [**4**] and Hanna Neumann [**7**], [**8**] obtained the bound $2(m-1)(n-1)+1$ for rank $A \cap B$. I have been unable to deduce from the preceding a bound for rank $A \cap B$ in terms of m and n, without using Howson's ideas.

2. Howson [**4**] remarked that if A and B both have finite index in F then it follows (by Schreier's formula) that

$$\text{rank } A \cap B \leqq (m - 1)(n - 1) + 1.$$

This is also true when only one of A and B has finite index in F. For suppose without loss of generality that B has finite index (j, say). The group F necessarily has finite rank (r, say). We may assume that $r \geqq 2$ since the case $r = 1$ is trivial. By Schreier's formula $n = j(r-1)+1$, whence $n \geqq j+1$. The index i of $A \cap B$ in A is the number of cosets of B containing elements of A and hence $i \leqq j \leqq n-1$. Thus by Schreier's formula, applied to $A \cap B$ in A,

$$\text{rank } A \cap B = i(m - 1) + 1 \leqq (n - 1)(m - 1) + 1.$$

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