QUASI-REFLEXIVITY AND DUAL NORMS¹

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A Banach space X with norm p will be denoted by (X, p) when it is important to emphasize the norm p . The *dual norm* p^* is defined on X^* by

$$p^*(f) = \sup\{|f(x)| : p(x) \leq 1\}.$$

If r is another norm on X equivalent to p , the duals $(X, p)^*$ and $(X, r)^*$ coincide and will both be denoted by X^* , and p^* and r^* are equivalent norms on X^* . Williams has shown [5] that (X, p) is reflexive if and only if every norm on X^* equivalent to p^* is the dual of some norm on X equivalent to p . The following theorem is a generalization of that result.

THEOREM. *A Banach space (X, p) is quasi-reflexive of order $\leq n$, where n is a nonnegative integer, if and only if X^* has a subspace S of codimension $\leq n$ such that for every norm q on X^* equivalent to p^* , there exists a norm t on X equivalent to p such that $t^*|_S = q|_S$.*

PROOF. (i) Suppose X is quasi-reflexive of order $\leq n$. By [1, p. 908] there exist a Banach space (Y, r) (quasi-reflexive of the same order as X) and a linear homeomorphism ϕ from (X, p) onto (Y^*, r^*) . Let J be the canonical map from Y into Y^{**} . Let $S = \phi^*[JY]$.

The codimension of S in X^* is equal to the codimension of JY in Y^{**} , which is $\leq n$. Suppose q is a norm on X^* equivalent to p^* . Define q_2 on Y^{**} by $q_2(F) = q(\phi^*F)$, where ϕ^* is the transpose of ϕ . Then q_2 and r^{**} are equivalent norms on Y^{**} since q and p^* are equivalent norms on X^* and since ϕ^* is a linear homeomorphism from (Y^{**}, r^{**}) onto (X^*, p^*) . Define q_3 on Y by $q_3(y) = q_2(Jy)$. Note that $r(y) = r^{**}(Jy)$. Now q_3 and r are equivalent norms on Y since q_2 and r^{**} are equivalent norms on Y^{**} and since J is a linear isomorphism from Y into Y^{**} . Thus q_3^* and r^* are equivalent norms on Y^* . Define t on X by $t(x) = q_3^*(\phi x)$. Then t and p are equivalent norms on X since ϕ is a linear homeomorphism from (X, p) onto (Y^*, r^*) . Finally, $t^*|_S = q|_S$ since for all $y \in Y$ we have

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$$\begin{aligned} t^*(\phi^*Jy) &= \sup\{ |(\phi^*Jy)x| : t(x) \leq 1 \} = \sup\{ |(Jy)(\phi x)| : q_s^*(\phi x) \leq 1 \} \\ &= q_s^{**}(Jy) = q_s(y) = q_2(Jy) = q(\phi^*Jy). \end{aligned}$$

(ii) Suppose that X^* has a subspace S of codimension $\leq n$ such that for every norm q on X^* equivalent to p^* , there exists a norm t on X equivalent to p such that $t^*|_S = q|_S$. Let J be the canonical map from X into X^{**} . We show that $X^{**} = JX + S^\perp$, from which it follows that X is quasi-reflexive of order $\leq n$ since $\dim S^\perp \leq \text{codim } S \leq n$ [4, Theorem 3, p. 39]. Let $F \in X^{**}$. Let $U = \{f \in X^*: p^*(f) \leq 1\}$. For each positive integer j , let $V_j = \{f \in X^*: |F(f)| \leq 1/j\} \cap U$. Each set V_j is a balanced, convex, p^* -bounded, p^* -closed, p^* -neighborhood of zero. Consequently V_j has a gauge q_j which is equivalent to p^* [4, p. 58]. Thus V_j is q_j -closed, and $V_j = \{f \in X^*: q_j(f) \leq 1\}$. By hypothesis there exists a norm t_j on X equivalent to p such that $t_j^*|_S = q_j|_S$. Let $U_j = \{f \in X^*: t_j^*(f) \leq 1\}$. Note that the $w(X^*, X)$ -topology on X^* is independent of the various norms under consideration. Each set U_j is $w(X^*, X)$ -closed by the Banach-Alaoglu Theorem [4, Theorem 1, p. 239]. We have $V_j \cap S = U_j \cap S$ since $q_j|_S = t_j^*|_S$. Therefore $V_j \cap S$ is relatively $w(X^*, X)$ -closed in S . Now $F^{-1}(0) \cap U \cap S = \bigcap_{j=1}^\infty (V_j \cap S)$, so $F^{-1}(0) \cap U \cap S$ is relatively $w(X^*, X)$ -closed in S . Let K be the $w(X^*, X)$ -closure in X^* of $F^{-1}(0) \cap U \cap S$, and let L be the linear span of K . Now $K \supseteq p^* - \text{cl}(F^{-1}(0) \cap U \cap S)$ since the $w(X^*, X)$ topology is weaker than the p^* topology. Also $p^* - \text{cl}(F^{-1}(0) \cap U \cap S) = U \cap p^* - \text{cl}(F^{-1}(0) \cap S)$ since $F^{-1}(0) \cap S$ is a subspace and since U is the p^* -unit ball. This $L \supseteq p^* - \text{cl}(F^{-1}(0) \cap S)$. Since $F^{-1}(0) \cap S$ has finite codimension in X^* , the subspace L is the direct sum of $p^* - \text{cl}(F^{-1}(0) \cap S)$ and some finite-dimensional subspace, and hence L is p^* -closed by [3, p. 59]. The set K is convex since it is the closure of a convex set [2, Theorem 1, p. 413]. It now follows by a corollary of the Krein-Šmulian Theorem [2, Corollary 9, p. 429] that L is $w(X^*, X)$ -closed. We have $L \cap S \supseteq F^{-1}(0) \cap S$ since $L \supseteq p^* - \text{cl}(F^{-1}(0) \cap S)$. Since K is convex and balanced, $L = \text{span } K = \{ak : k \in K \text{ and } a > 0\}$. Thus

$$L \cap S \subseteq \text{span}(K \cap S) = \text{span}(F^{-1}(0) \cap U \cap S) = F^{-1}(0) \cap S.$$

We now have $F^{-1}(0) \cap S = L \cap S$, which is relatively $w(X^*, X)$ -closed. Thus $F|_S$ is $w(S, X)$ -continuous by [4, Theorem 3, p. 186].

By [3, Theorem 14.1, pp. 117–118], there exists a $w(X^*, X)$ -continuous linear functional G on X^* such that $G|_S = F|_S$. Now $F = G + (F - G) \in JX + S^\perp$, and we have $X^{**} = JX + S^\perp$, which completes the proof.

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