A NOTE ON CARDINAL REFLECTIONS IN THE CATEGORY OF UNIFORM SPACES

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In [5, Chapter iii, Exercise 2] it is shown that the countable uniform covers of any uniform space form a uniformity, and it is asked whether the same thing happens for arbitrary infinite cardinals. Here a partial affirmative answer is given: when its uniformity has a basis of σ -point-finite uniform covers (a family of sets is said to be σ -point-finite iff it is a union of countably many subfamilies each of which is a point-finite family), the space admits arbitrary cardinal reflections. This means that a positive solution of [5, Research Problem B₂] implies a positive solution of the above question. But the major interest of our result is that it furnishes the expected and most natural characterization of the uniformities defined in [1, §3] by extended reticles: they are the cardinal reflections of the fine uniformity of the given space. From this, a very easy proof of the main results of [2] is derived, and it seems that almost all the work of G. Aquaro in that field may be considerably simplified at least in the technical apparatus (this is certainly true for [3]).

Our terminology and notations are based on [5] (but we do not assume *a priori* the axiom T_2 on uniform spaces).

1. Main results. Note that the uniformity ν of the following theorem is compatible with the topology of X because it is finer than the precompact reflection $p\mu X$.

1.1 THEOREM. *Let μX be a uniform space and $\aleph$ any infinite cardinal. If μ has a basis of σ -point-finite coverings, then the uniform covers of μX with cardinality $\leq \aleph$ form a uniformity ν on X .*

PROOF. We must prove that every uniform covering of μX of cardinality $\leq \aleph$ has a uniform star-refinement of cardinality $\leq \aleph$. We begin by establishing the following result:

(*) Every $\mathfrak{U} \in \mu$ of cardinal $\leq \aleph$ has a uniform refinement which is σ -point-finite and has cardinal $\leq \aleph$.

By hypothesis, there is a uniform refinement $(V_{n\alpha})_{n\alpha}$ of $\mathfrak{U}$ such that, for each $n \in \mathbb{Z}^+$, $(V_{n\alpha})_{\alpha \in A_n}$ is point-finite. Let $\leq$ be a well-ordering for $\mathfrak{U}$ and, for each $n \in \mathbb{Z}^+$ and each $U \in \mathfrak{U}$, let $A_U^n = \{\alpha \in A_n \mid V_{n\alpha} \subseteq U\}$. Obviously the sets

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$$B_U^n = A_U^n \setminus \bigcup \{ A_{U'}^n \mid U' \in \mathfrak{U}, U' < U \} \quad (U \in \mathfrak{U})$$

form a partition of A_n . Define $W_{nU} = \bigcup \{ V_{n\alpha} \mid \alpha \in B_U^n \}$. Then $(W_{nU})_{nU}$ refines $\mathfrak{U}$, is refined by $(V_{n\alpha})_{n\alpha}$ and hence is a uniform cover of μX , and its index set $\mathbf{Z}^+ \times \mathfrak{U}$ has cardinality $\leq \aleph$. Finally, for each $n \in \mathbf{Z}^+$, $(W_{nU})_{U \in \mathfrak{U}}$ is point-finite: to any $x \in X$ there corresponds a finite set $\{\alpha_1, \dots, \alpha_m\} \subseteq A_n$ such that $\{V_{n\alpha_1}, \dots, V_{n\alpha_m}\}$ is exactly the set of all $V_{n\alpha}$'s which contain x . Then x lies exactly in the members of the finite subfamily $(W_{nU_1}, \dots, W_{nU_m})$ of $(W_{nU})_U$, where U_i is defined by $\alpha_i \in B_{U_i}^n$. Thus (*) holds.

The following reasoning is largely inspired to the proof of [4, Theorem 1.2]. Choose $\mathfrak{U} \in \mu$ with cardinal $\leq \aleph$. By (*), $\mathfrak{U}$ has a uniform refinement $(V_{n\alpha})_{n\alpha}$ such that, for each $n \in \mathbf{Z}^+$, $(V_{n\alpha})_{\alpha \in A_n}$ is point-finite and $\text{Card}(A_n) \leq \aleph$. Let $\mathfrak{J} = (T_\lambda)_\lambda \in \mu$ be such that $\mathfrak{J}^{**}$ refines $(V_{n\alpha})_{n\alpha}$ and $T_\lambda \neq \emptyset$ for all λ . Define:

$$\begin{aligned} m(\lambda) &= \min \{ n \in \mathbf{Z}^+ \mid \text{St}(T_\lambda, \mathfrak{J}) \subseteq V_{n\alpha} \text{ for some } \alpha \in A_n \}; \\ n(\lambda) &= \min \{ n \in \mathbf{Z}^+ \mid \text{St}(\text{St}(T_\lambda, \mathfrak{J}), \mathfrak{J}) \subseteq V_{n\alpha} \text{ for some } \alpha \in A_n \}; \\ q(\lambda) &= \{ (n, \alpha) \mid n \leq n(\lambda), V_{n\alpha} \supseteq T_\lambda \}. \end{aligned}$$

Because every T_λ is nonempty and every $(V_{n\alpha})_\alpha$ is point-finite, $q(\lambda)$ is a finite subset of $\mathbf{Z}^+ \times \bigcup_{n=1}^\infty A_n$. For all $m, n \in \mathbf{Z}^+$ and all finite subsets F of $\mathbf{Z}^+ \times \bigcup_{n=1}^\infty A_n$, define

$$W_{mnF} = \bigcup \{ T_\lambda \mid m(\lambda) = m, n(\lambda) = n, q(\lambda) = F \}.$$

Clearly $(W_{mnF})_{mnF}$ has cardinality $\leq \aleph$ and is refined by $\mathfrak{J}$, so that it is a uniform cover of μX . For every λ_0 there is $\alpha(\lambda_0) \in A_{m(\lambda_0)}$ such that $\text{St}(T_{\lambda_0}, \mathfrak{J}) \subseteq V_{m(\lambda_0)\alpha(\lambda_0)}$. If $W_{mnF} \cap W_{m(\lambda_0)n(\lambda_0)q(\lambda_0)} \neq \emptyset$, then there is a λ such that $m(\lambda) = m, n(\lambda) = n, q(\lambda) = F$ and $T_\lambda \cap T_{\lambda_0} \neq \emptyset$. It follows that $T_{\lambda_0} \subseteq \text{St}(T_\lambda, \mathfrak{J})$ and $\text{St}(T_{\lambda_0}, \mathfrak{J}) \subseteq \text{St}(\text{St}(T_\lambda, \mathfrak{J}), \mathfrak{J}) \subseteq V_{n(\lambda)\alpha(\lambda)}$ for a suitable $\alpha(\lambda) \in A_{n(\lambda)}$. Therefore $m(\lambda_0) \leq n(\lambda)$. From this and $T_\lambda \subseteq \text{St}(T_{\lambda_0}, \mathfrak{J}) \subseteq V_{m(\lambda_0)\alpha(\lambda_0)}$ it follows that $(m(\lambda_0), \alpha(\lambda_0)) \in q(\lambda) = F$, which implies $W_{mnF} \subseteq V_{m(\lambda_0)\alpha(\lambda_0)}$: hence $(W_{mnF})_{mnF}$ is a uniform star-refinement of $\mathfrak{U}$. Q.E.D.

Now we desire to recall the definition—given in [1, §3]—of what G. Aquaro calls $\aleph$ -uniformity of a given topological space X , $\aleph$ being an infinite cardinal. Because the original definition is fairly complicated, we prefer to give an equivalent definition (the equivalence follows from [1, §2, Theorem 1-(e)]) which is more closely related to our purposes: The $\aleph$ -uniformity of X has a base of uniform coverings consisting of all families $(U_i)_{i \in I}$ such that $\text{Card}(I) \leq \aleph$, $(U_i)_i$ is a locally finite open cover of X and there is a closed covering $(F_i)_{i \in I}$ of X and an *entourage* W of a uniformity whose topology is less fine

than the topology of X for which $W[F_i] \subseteq U_i$ holds for all $i \in I$. Recall that the *fine uniformity* of a uniformizable space is the finest uniformity compatible with its topology.

The following result—which may be read alternatively: the $\aleph_\alpha$ -uniformity of X is the $\aleph_{\alpha+1}$ -reflection of the fine uniformity of X —also implies the same conclusion when X is not assumed uniformizable, as in Aquaro's works: it suffices to apply it to the weak topology determined by the set of all continuous mappings $X \rightarrow \mathbb{R}$.

1.2. COROLLARY. *For every uniformizable space X , the $\aleph$ -uniformity of X is exactly the uniformity having as basis the set of all uniform coverings of the fine uniformity of X which have cardinal $\leq \aleph$.*

PROOF. Because the fine uniformity μ_∞ of X has a basis of uniformly locally finite uniform coverings by [5, vii.4] (or, alternatively, by the result of [1]), 1.1 implies that the set of all uniform coverings of μ_∞ having cardinal $\leq \aleph$ is really the base for a uniformity μ on X (this is the sole application of 1.1). Let ν be the $\aleph$ -uniformity of X . By the definition of ν given above (since $\{W[x] \mid x \in X\}$ are uniform covers whenever W is an *entourage* of the given uniformity), it is clear that ν is less fine than μ . Conversely, assume $\mathfrak{U} \in \mu$ and $\text{Card}(\mathfrak{U}) \leq \aleph$. Because μ_∞ has a base of locally finite uniform covers, we may repeat with obvious changes (it suffices to delete “ n ” whenever it appears) the proof of statement (*) in the proof of 1.1 to obtain a refinement $\mathfrak{V}$ of $\mathfrak{U}$ such that $\mathfrak{V} \in \mu_\infty$, $\text{Card}(\mathfrak{V}) \leq \aleph$ and $\mathfrak{V}$ is an open locally finite cover of X . Let $\mathfrak{J} \in \mu_\infty$ be an open star-refinement of $\mathfrak{V}$. For any $V \in \mathfrak{V}$, define

$$\mathfrak{J}_V = \{T \in \mathfrak{J} \mid \text{St}(T, \mathfrak{J}) \subseteq V\}; \quad Z_V = \bigcup \{T \mid T \in \mathfrak{J}_V\}.$$

Since each element of $\mathfrak{J}$ is open, $\text{St}(\bar{Z}_V, \mathfrak{J}) \subseteq \bigcup \{\text{St}(T, \mathfrak{J}) \mid T \in \mathfrak{J}_V\} \subseteq V$. This implies the existence of an *entourage* W of $\mu_\infty X$ such that $W[\bar{Z}_V] \subseteq V$ for all $V \in \mathfrak{V}$. Q.E.D.

2. Applications to G. Aquaro's works. The aim of this section is to show how 1.2 may be used to simplify some proofs of [2]: the best result is (b), the simplification of the original proof of [2, Proposition 7] avoiding the use of the very complicated [2, Proposition 6]. Also the proofs of [3, Lemmas 2 and 3, Proposition 6] may be simplified, but we examine only [2] because the simplifications of [3] are not so spectacular.

Recall that, according to [2], a uniformizable space X is said to be $\aleph$ -complete, with $\aleph$ an infinite cardinal, iff the $\aleph$ -uniformity of X is complete.

(a) [2, PROPOSITION 5]. *Any product of $\aleph_N$ -complete spaces is $\aleph_N$ -complete.*

PROOF. By [5, Chapter ii, Exercise 2], the product of uniformities with a base of uniform covers with cardinal $\leq \aleph_N$ has a base of covers with cardinality $\leq \aleph_N$. Then the conclusion follows from 1.2 and the fact that every uniformity finer than a complete one is complete. Q.E.D.

(b) [2, PROPOSITION 7]. *If μ is the $\aleph_N$ -uniformity of X , then there is an $\aleph_N$ -complete space Y such that μX is isomorphic to a dense subspace of νY , ν being the $\aleph_N$ -uniformity of Y .*

PROOF. Let Y be the completion of μX . By [5, ii.9], $\{\bar{U} \mid U \in \mathfrak{u}\}$ is a uniform cover of Y for all $\mathfrak{u} \in \mu$. By this and 1.2, the uniformity of Y is less fine than ν , the $\aleph_N$ -uniformity of Y . Thus ν is complete and, again by 1.2, induces on X exactly μ . Q.E.D.

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