THE RANGE OF A VECTOR-VALUED MEASURE

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Liapounoff, in 1940, proved that the range of a countably additive bounded measure with values in a finite dimensional vector space is compact and, in the nonatomic case, is convex. Later, in 1945, Liapounoff showed, by counterexample, that neither the convexity nor compactness need hold in the infinite dimensional case. The next step was taken by Halmos who in 1948 gave simplified proofs of Liapounoff's results for the finite dimensional case. In 1951, Blackwell [1] considered the case of a measure represented by a finite dimensional vector integral and obtained results similar to those of Liapounoff for these measures. Various versions of Liapounoff's theorem appeared in the 1950's and 1960's, and in 1966, Lindenstrauss [8] gave a very elegant short proof of Liapounoff's earlier result. Finally, in 1968, Olech [9] considered the case of an unbounded measure with range in a finite dimensional vector space. The purpose of this note is to demonstrate that the closure of the range of a measure of bounded variation with values in a Banach space, which is either a reflexive space or a separable dual space, is compact and, in the nonatomic case, is convex.

To this end, let Ω be a point set and Σ be a σ-field of subsets of Ω. If $\mathfrak{X}$ is a Banach space, then an $\mathfrak{X}$-valued measure is a countably additive function F defined on Σ with values in $\mathfrak{X}$. F is of bounded variation if

$$\mathrm{var}(F)(\Omega) = \sup_{\pi} \sum_{n} \| F(E_n) \| < \infty$$

where the supremum is taken over all partitions $\pi = \{E_n\}_{n=1}^{m} \subset \Sigma$ consisting of a finite collection of disjoint sets in Σ whose union is Ω. A set $E \in \Sigma$ is an atom of F if $F(E) \neq 0$ and $E' \in \Sigma$, $E' \subset E$ imply $F(E') = 0$ or $F(E') = F(E)$. F is nonatomic if F has no atoms.

The following theorem is the main result of this note.

THEOREM 1. *Let $\mathfrak{X}$ be a Banach space which is either a reflexive space or a separable dual space. If $F: \Sigma \to \mathfrak{X}$ is a measure of bounded variation, then the range of F is a precompact set in the norm topology of $\mathfrak{X}$. Moreover, if F is nonatomic, then the closure of the range of F is compact and convex.*

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PROOF. Let F and $\mathfrak{x}$ be as in the hypothesis, and for $E \in \Sigma$, let $\mu(E)$ be the variation of F restricted to E. (I.e., $\mu(E) \neq S$ is the variation of the set function $F \cdot E(\cdot) = F(E \cap \cdot)$ on Σ.) Then according to Dinculeanu [2, p. 41], μ is a countably additive nonnegative finite measure on Σ. Clearly F is absolutely continuous with respect to μ. Hence, in the case $\mathfrak{x}$ is reflexive or in the case $\mathfrak{x}$ is a separable dual space, Phillips' generalization of the Radon-Nikodým theorem [10, p. 30] or the Dunford and Pettis theorem [4, Theorem 2.1.4], respectively, guarantee the existence of a μ-measurable $\mathfrak{x}$-valued function $f \in L^1(\mu, \mathfrak{x})$ (i.e., $\int_\Omega \|f\| d\mu < \infty$) such that $F(E) = \int_E f d\mu$ for all $E \in \Sigma$.

Next, select a sequence of simple functions $\{f_n\}$ in $L^1(\mu, \mathfrak{x})$ converging to f in $L^1(\mu, \mathfrak{x})$ norm and define T and T_n, $n = 1, 2, \cdots$, for $g \in L^\infty(\mu, \mathbf{C})$ ($\mathbf{C}$ = scalar field of $\mathfrak{x}$) by $T(g) = \int_\Omega gf d\mu$ and $T_n(g) = \int_\Omega gf_n d\mu$ respectively. Then T and T_n are evidently linear and by the Hölder inequality,

$$\left\| \int_\Omega gf d\mu \right\| \leqq \int_\Omega |g| \, \|f\| d\mu \leqq \|g\|_{L^\infty} \|f\|_{L^1},$$

are bounded. In addition the last computation shows that, in the uniform operator topology, $\lim_n \|T_n - T\| \leqq \lim_n \int_\Omega \|f - f_n\| d\mu = 0$. Now, note that the range of each T_n is finite dimensional since each f_n is a simple function. Therefore each T_n is compact, and hence, by the above, $T: L^\infty(\mu, \mathbf{C}) \to \mathfrak{x}$ is a compact operator. Moreover, since $\{\chi_E : E \in \Sigma\}$ is contained in the unit ball of $L^\infty(\mu, \mathbf{C})$, it follows from the compactness of T that

$$\{F(E) : E \in \Sigma\} = \left\{ \int_E f d\mu : E \in \Sigma \right\} = \{T(\chi_E), E \in \Sigma\}$$

is a norm precompact set in $\mathfrak{x}$. This proves the first assertion.

To prove the second statement, assume that F is nonatomic. Clearly μ, as defined above, is also nonatomic. Now, if $\pi = \{E_n\}$ is a partition and f_π is the simple function defined by

$$f_\pi = \sum_\pi \frac{\int_{E_n} f d\mu}{\mu(E_n)} \cdot \chi_{E_n},$$

$(0/0) = 0$ and F_π is the indefinite integral of f_π, i.e., for $E \in \Sigma$,

$$F_\pi(E) = \sum_\pi \frac{\int_{E_n} f d\mu}{\mu(E_n)} \mu(E_n \cap E),$$

then by [3, Theorem III.2.15] and [3, Theorem IV.8.18],

$$\lim_{\pi} \operatorname{var}(F - F_\pi) = \lim_{\pi} \int_\Omega \|f - f_\pi\| d\mu = 0,$$

where the limit is taken in the Moore-Smith sense after the collection of all partitions is directed by the partial ordering of refinement.

Next note that each of the $\mathfrak{X}$-valued measures F_π has its values in a finite dimensional subspace of $\mathfrak{X}$. Also, since μ is nonatomic, it follows easily that each F_π is nonatomic and hence by Liapounoff's theorem [5] has a convex range. Now let x, y belong to the closure of the range of F, α and β be nonnegative numbers with $\alpha+\beta=1$, and $\epsilon>0$ be given. Select E_1 and $E_2 \in \Sigma$ such that $\|x-F(E_1)\|<\epsilon/2$ and $\|y-F(E_2)\|<\epsilon/2$. Then choose a partition π_0 subject to the conditions that

$$\pi_0 \geqq \{E_1 - E_2, E_2 - E_1, E_1 \cap E_2, \Omega - (E_1 \cup E_2)\}$$

and that $\operatorname{var}(F-F_{\pi_0})<\epsilon/2$. It is not difficult to see that

$$F_{\pi_0}(E_i) = \int_{E_i} f d\mu = F(E_i), \qquad i = 1, 2, \cdots$$

Moreover, since the range of F_{π_0} is convex, there exists a set $E_0 \in \Sigma$ such that $F_{\pi_0}(E_0)=\alpha F_{\pi_0}(E_1)+\beta F_{\pi_0}(E_2)=\alpha F(E_1)+\beta F(E_2)$. Combining these relationships, one has

$$\begin{aligned} &\|\alpha x + \beta y - F(E_0)\| \\ &\quad = \|\alpha x + \beta y - (\alpha F(E_1)) + \beta(F(E_2)) + F_{\pi_0}(E_0) - F(E_0)\| \\ &\quad \leqq \alpha\|x - F(E_1)\| + \beta\|y - F(E_2)\| + \|F_{\pi_0}(E_0) - F(E_0)\| \\ &\quad < \alpha\epsilon/2 + \beta\epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

since $\alpha+\beta=1$ and $\|F_{\pi_0}(E_0)-F(E_0)\| \leqq \operatorname{var}(F_{\pi_0}-F)<\epsilon/2$. Thus the closure of the range F is convex. Q.E.D.

The following corollary is clear.

COROLLARY 2. *Under the same hypothesis, if the range of F is closed, then it is norm compact. If F is nonatomic and its range is closed, then its range is compact and convex.*

Neither the theorem nor its corollary have immediate improvements. Below are two examples, the first indicates that if the hypothesis on $\mathfrak{X}$ is weakened, the conclusion of Theorem 1 fails, and the second, which is due to Liapounoff, shows that a measure may satisfy the hypothesis of Theorem 1 and fail to have a compact or convex range.

First, let $\Omega=[0, 1]$, Σ be the Borel σ-field of subsets of Σ and λ be Lebesgue measure on Σ. Define $F: \Sigma \rightarrow L^1(\Omega, \Sigma, \mu)$ by $F(E)=\chi_E$, where χ_E is the characteristic or indicator function of $E\in\Sigma$. Clearly F is nonatomic, and since $\|F(E)\|=\|\chi_E\|=\lambda(E)$, F is evidently countably additive and of bounded variation. It will now be shown that the closure of the range of F is neither compact nor convex. To show the range of F is not precompact, consider the Borel sets

$$E_m = \bigcup_{n=1}^{2^{(m-1)}} E_{mn}, \qquad m = 1, 2, \cdots$$

where E_{mn} is the closed interval $[2(n-1)/2^m, (2n-1)/2^m]$ for $n=1, 2, \cdots, 2^{(m-1)}$. A brief computation yields $\|\chi_{E_i}-\chi_{E_j}\|=1/4$ for $i\neq j$. Thus $\{\chi_{E_m}\}=\{F(E_m)\}$ is a sequence in the range of F with no convergent subsequence; i.e., the range of F is not precompact.

To show that the closure of the range of F is not convex, note that the function

$$1/2\chi_\Omega = 1/2\chi_{E_1} + 1/2\chi_{E_2}$$

where $E_1=[0, 1/2]$ and $E_2=[1/2, 1]$ is a convex combination of members of the range of F. But, if $E\in\Sigma$ is arbitrary

$$\|F(E) - 1/2\chi_\Omega\| = \|\chi_E - 1/2\chi_\Omega\| = 1/2\lambda(\Omega - E) + 1/2\lambda(E) = 1/2.$$

Thus the closure of the range of F is not convex.

Remark. It is noted here that in view of Theorem 1, this example provides another proof of the fact that the separable space $L^1(\Omega, \Sigma, \mu)$ is not a dual space. Also this example provides a simple set function F absolutely continuous with respect to λ but which has no Radon-Nikodým derivative with respect to λ. To see this, note that if F were an integral with respect to λ then the proof of Theorem 1 would show that the range of F were compact.

Finally the example constructed by Liapounoff in [6] will be given with a minor modification to show that even if a vector measure F satisfies the hypothesis of Theorem 1, then its range need not be compact or convex. It is given here for completeness and because of the one small modification. Let $[0, 2\pi]=\Omega$, Σ be the Borel σ-field of subsets of Ω, and λ be Lebesgue measure on Σ. Let $\{\psi_n\}_{n=0}^{\infty}$ be a complete orthogonal set in $L^2(\lambda, \mathbf{C})$ such that each ψ_i assumes only the values $+1$ and -1 and such that $\psi_0\equiv +1$ while $\int_0^{2\pi}\psi_n d\lambda=0$ for $n>0$.[1] Defining I_n on Σ by

[1] Any normalized Haar basis will suffice.

$$I_n(E) = 2^{-n}\int_E ((1+\psi_n)/2)d\lambda, \qquad E\in\Sigma$$

and $F\colon E\to l^2$ by

$$F(E) = (I_0(E), I_1(E), \cdots, I_n(E), \cdots),$$

one finds $\|F(E)\|_{l^2}\leqq 2\,\lambda(E)$ so shat F is of bounded variation. Clearly F is nonatomic; therefore since F has its values in the reflexive space l^2 Theorem 1 guarantees the closure of the range of F is compact and convex. Now consider $F(\Omega)=(2\pi,\ \pi/2,\ \pi/4,\ \cdots,\ \pi/2^n,\ \cdots)$ and suppose there exists an $E\in\Sigma$ such that $F(E)=F(\Omega)/2$. Then $\pi=I_0(E)=\int_E d\lambda=\lambda(E)$ and for $n>0$

$$\pi/2^{n+1} = I_n(E) = 2^{-n}\int_E ((1+\psi_n)/2)d\lambda$$
$$= \lambda(E\cap U_n)/2^n$$

where $U_n=\{s\in[0,\ 2\pi]:\psi_n(s)=+1\}$. It follows immediately from this and the facts that $\lambda(U_n)=\lambda(E)=\pi$ that $\lambda(E\cap U_n)=\lambda(E-U_n)=\lambda(U_n-E)=\lambda(-E-U_n)=\pi/2$ for all $n>0$. Now define ω on Ω by $\omega(x)=+1$ for $x\in E$, $\omega(x)=-1$ for $x\notin E$. Then $\int_0^{2\pi}\psi_0\omega d\lambda=\pi-\pi=0$, and for $n>0$

$$\int_0^{2\pi}\psi_n\omega d\lambda = \lambda(U_n\cap E)+\lambda(-U_n-E)$$
$$-\lambda(E-U_n)-\lambda(E-U_n) = 0.$$

This contradicts the fact that $\{\psi_n\}$ was complete in $L^2(\lambda,\ \mathbf{C})$ and shows two things: first, that even under the hypothesis of Theorem 1, such a measure F need not have a convex range and second, that in view of Corollary 2 that the range of such a measure need not be closed. Thus Theorem 1 cannot be improved under the current hypothesis.

It would be interesting to remove the restrictions imposed by Theorem 1 on the range space $\mathfrak{x}$. If $\mathfrak{x}$ is allowed to be a general Banach space and F is an $\mathfrak{x}$-valued measure of bounded variation, then one can assert that the range of F is precompact and that, in the nonatomic case the closure of the range of F is convex if, as the proof of Theorem 1 shows, F has the representation $F(E)=\int_E f d\mu$, $E\in\Sigma$ for some measure μ and some measurable f with $\int_\Omega\|f\|d\mu<\infty$. However, this restriction appears, to the author, to be too severe for a general result.

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