A SUMMABLE C_Ω -GROUP

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Recently, Megibben [3] has enriched, in a sense manifold, the theory of torsion abelian groups by developing a " C_λ -theory" for each countable limit ordinal λ . The classical theory of torsion abelian groups corresponds to Megibben's C_ω -theory. Let p be a fixed prime. A p -primary group G belongs to the class C_λ , for any ordinal λ , if and only if $G/p^\alpha G$ is a direct sum of countable groups (d.s.c.) for each $\alpha < \lambda$. One should note that C_ω is the whole class of primary groups and that $C_\mu \subseteq C_\lambda$ is $\lambda \leq \mu$. The question of to what extent the C_λ -theorems for a countable limit λ carry over to C_Ω -theorems, where Ω is the first uncountable ordinal, has been only partially settled. As we have already observed, $C_\Omega \subseteq C_\lambda$ but the point is C_Ω -theorems are stronger than C_λ -theorems for $\lambda < \Omega$. In particular, we have the following question. Can the countability condition on λ be removed in (either of) the next two theorems?

THEOREM (HILL AND MEGIBBEN [2]). *Let λ be a countable limit ordinal. If G is a summable C_λ -group of length λ , then G is a d.s.c.*

THEOREM (MEGIBBEN'S CRITERION). *Let λ be a countable limit ordinal. Let G be a summable, p -primary group of length λ . If, for each $\alpha < \lambda$, G contains a p^α -high subgroups which is a d.s.c., then G itself is a d.s.c.*

For the special case $\lambda = \omega$, Megibben's criterion is essentially the same as Kulikov's criterion, see [4]. But, of course, Megibben's theorem has much broader scope. The purpose of this paper is to show, however, that this scope does not encompass C_Ω -groups.

THEOREM. *A summable C_Ω -group need not be a d.s.c.*

PROOF. The proof is constructive. For each countable ordinal α let G_α be the unique countable, primary group of length α having each of its nonzero Ulm invariants equal to $\aleph_0$. Suppose that $\gamma < \Omega$ and that we have embedded G_α in G_β , for each $\alpha < \beta < \gamma$, in such a way that

$$(*) \quad p^\lambda G_\beta \cap G_\alpha = p^\lambda G_\alpha \quad \text{for all } \lambda$$

holds. If γ is a limit ordinal, then $G_\gamma \cong \bigcup_{\alpha < \gamma} G_\alpha$ and we have an embedding of G_α in G_γ such that $(*)$ holds for $\alpha < \beta \leq \gamma$. If γ is a nonlimit ordinal, we distinguish two cases.

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Case 1: $\gamma - 1$ is a limit. Let $G_{\gamma-1}$ be a $p^{\gamma-1}$ -high subgroup of G_γ .

Case 2. $\gamma - 2$ exists. Let $G_{\gamma-1}$ be a direct summand of G_γ .

We remark, in connection with Case 1, that any $p^{\gamma-1}$ -high subgroup of G_γ is isomorphic to $G_{\gamma-1}$. As far as Case 2 is concerned, $G_\gamma \cong G_{\gamma-1} + G_\gamma$. It is well known that $p^\lambda G_\gamma \cap G_{\gamma-1} = p^\lambda G_{\gamma-1}$ for either type of embedding described above, so condition (*) continues to hold for $\alpha < \beta \leq \gamma$.

The group of interest is the union (=direct limit) of the groups G_α , as α ranges over the countable ordinals, embedded in one another in the precise manner described above. Let G denote this group, $G = \bigcup_{\alpha < \Omega} G_\alpha$. We wish to prove three things about G ; we want to show that G is summable, that $G/p^\alpha G$ is a d.s.c. for each countable α , and that G itself is not a d.s.c.

In order to show that G is summable, define $S_\alpha = p^\alpha G_{\alpha+1}[p]$ if α is a limit. If α is not a limit, let $G_{\alpha+1} = G_\alpha + K_\alpha$ and define $S_\alpha = K_\alpha[p]$. In either case, $G_{\alpha+1}[p] = G_\alpha[p] + S_\alpha$ is a *natural* decomposition [2] of $G_{\alpha+1}[p]$ in G —the decomposition honors heights computed in G . Thus $G[p] = \sum_{\alpha < \Omega} S_\alpha$ is a natural decomposition of all of $G[p]$. Since S_α is countable for each α , G is summable [2, Lemma 1.11].

In order to show that $G/p^\lambda G$ is a d.s.c. for each countable ordinal λ , we observe that the chain $\{G_\alpha, p^\lambda G\}/p^\lambda G$ of countable subgroups of $G/p^\lambda G$ leads up to $G/p^\lambda G$ and we analyze how these subgroups are embedded in one another. We claim that for $\alpha > \lambda$ — λ is fixed—that any such subgroup splits out of the succeeding one, thereby giving a decomposition of $G/p^\lambda G$ into a direct sum of countable groups. In this connection, observe that $\{G_\beta, p^\lambda G\}/p^\lambda G = \bigcup_{\alpha < \beta} (\{G_\alpha, p^\lambda G\}/p^\lambda G)$ if β is a limit. Now we proceed to back up our claim. Suppose that $\alpha > \lambda$. There are, of course, two cases to consider in showing that $\{G_\alpha, p^\lambda G\}/p^\lambda G$ is a direct summand of $\{G_{\alpha+1}, p^\lambda G\}/p^\lambda G$.

Case 1. α is a limit ordinal. Recall that in this case G_α is a p^α -high subgroup of $G_{\alpha+1}$. Thus $\{G_\alpha, p^\lambda G_{\alpha+1}\} = G_{\alpha+1}$ since $\lambda < \alpha$, and $\{G_\alpha, p^\lambda G\} = \{G_{\alpha+1}, p^\lambda G\}$. In particular, $\{G_\alpha, p^\lambda G\}/p^\lambda G$ is a direct summand of $\{G_{\alpha+1}, p^\lambda G\}/p^\lambda G$; the two are equal.

Case 2. α is not a limit. In this case, $G_{\alpha+1} = G_\alpha + K_\alpha$. It follows at once that

$$\{G_{\alpha+1}, p^\lambda G\}/p^\lambda G = \{G_\alpha, p^\lambda G\}/p^\lambda G + \{K_\alpha, p^\lambda G\}/p^\lambda G.$$

This completes the argument that $G/p^\lambda G$ is a d.s.c. for each countable λ . Thus, by definition, G is a C_Ω -group.

Finally, we show that G is not a d.s.c. Assume that G is a direct sum of countable groups; let $G = \sum_{i \in I} A_i$ where A_i is countable for each $i \in I$. It is easy to show, by the back-and-forth method of [2],

the existence of a countable subset J of I and a countable limit ordinal α such that $G_\alpha = \sum_{j \in J} A_j$. However, this is impossible since G_α is not a direct summand of G ; G_α is not even a direct summand of $G_{\alpha+1}$ since G_α is p^α -high in $G_{\alpha+1}$. We conclude that G is not a d.s.c. and the theorem is proved.

COROLLARY. *Megibben's criterion does not apply to groups of length Ω .*

PROOF. According to Hill's theorem [1] that any isotype subgroup of a d.s.c. of countable length is a d.s.c., any summable C_Ω -group satisfies Megibben's criterion. Hence the above theorem yields the corollary.

REFERENCES

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