EXTENSIONS OF TORSIONFREE GROUPS BY TORSION GROUPS

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1. It is well known and easily seen that every abelian group is an extension of a torsionfree group by a torsion group. K. A. Hirsch has proved in his paper quoted in the references at the end of this note, that the same is true for polycyclic groups (more precisely: since polycyclic torsion groups are finite, polycyclic groups are extensions of torsionfree groups by finite groups). But, of course, in general no such theorem can be expected as may be seen from the following example of a metabelian group G which is not an extension of a torsionfree group by a torsion group.

EXAMPLE, Let $E=\prod_{n=-\infty}^{\infty}\{a_n\}$ where $o(a_n)=p$, p a prime. Consider an element g such that $g^{-1}a_ng=a_{n+1}$ for all n. Then $G=E\{g\}$ contains the normal subgroup $E=\mathbf{C}(E)$ and E is an elementary abelian p-group. Since $E=\mathbf{C}(E)$, 1 is the only torsionfree normal subgroup of G. Evidently, $G''=1$ and G is neither a torsion group nor torsionfree.

Thus it will be interesting to investigate the intrinsic ideas underlying the Theorem of Hirsch and to find generalizations of it; this is the goal of the present paper.

2. **Notation.**

$\mathbf{C}(S)$ = centralizer of the subgroup S of G in G.

$x\circ y=x^{-1}y^{-1}xy$.

$A\circ B$ = subgroup generated by $a\circ b$ for all $a\in A$, $b\in B$.

$o(x)$ = order of the element x.

If $\mathfrak{e}$ is a group-theoretical property, then an $\mathfrak{e}$-group is a group with property $\mathfrak{e}$.

noetherian = each subgroup is finitely generated.

hyperabelian = each epimorphic image $H\neq 1$ contains an abelian normal subgroup $\neq 1$.

solvable i.e. $G^{(k)}=1$ for suitable $k\geqq 0$.

polycyclic = noetherian and hyperabelian = noetherian and solvable.

hypercentral = each epimorphic image $H\neq 1$ has a nontrivial center.

If G is hypercentral then there exists a smallest ordinal α such that $Z_\alpha=G$, where $Z_0=1$, $Z_1=Z(G)$ = center of G, if λ is a limit ordinal $Z_\lambda=\bigcup_{\beta<\lambda}Z_\beta$, and $Z_{\beta+1}/Z_\beta=Z(G/Z_\beta)$. Furthermore, α is called the

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class of G. If α is finite, G is also called nilpotent. The chain $1=Z_0 \leqq Z_1 \leqq \cdots$ is called the ascending central chain of G.

$N \trianglelefteq G$ i.e. N is normal in G; $N \triangleleft G$ i.e. $N \trianglelefteq G$ and $N \neq G$. $\prod$ = direct product.

n-group = a group all of whose elements have order a power of n.

3. If a group G is the extension of a torsionfree group by a torsion-group, then the same is true for all subgroups of G but not necessarily for all epimorphic images of G. Since all epimorphic images of abelian (polycylic) groups are abelian (polycyclic), each epimorphic image of an abelian (polycyclic) group is the extension of a torsionfree group by a torsion group. Therefore we will begin our discussion with the investigation of those groups G whose epimorphic images are the extension of a torsionfree group by a torsion group and we will prove the equivalence of the following three conditions:

(A) Each epimorphic image H of G is an extension of a torsionfree group by a torsion group.

(B) (i) If T is a normal torsion subgroup of the epimorphic image H of G, then $H/\mathbf{C}(T)$ is a torsiongroup. (ii) If the epimorphic image H of G is not a torsiongroup, if 1 is the only torsionfree normal subgroup of H and if P is the maximal normal torsion subgroup of H, then $Z(P)=1\neq P$.

(C) (i) If T is a normal torsion subgroup of the epimorphic image H of G, then there exists an epimorphic torsion image H^* of G containing $T^* \trianglelefteq H^*$ such that $T^* \cong T$ and $H/\mathbf{C}(T) \cong H^*/\mathbf{C}(T^*)$. (ii) If $A\neq 1$ is an abelian normal torsion subgroup of the epimorphic image H of G and H/A is the extension of a torsionfree group by a torsion group, then H is the extension of a torsionfree group by a torsion group. (iii) If the epimorphic image H of G is not the extension of a torsionfree group by a torsion group, then H is the extension of a torsion group by a torsionfree group.

An inspection of the proof of the equivalence of (A), (B) and (C) will reveal that one can formalize the proof using group-theoretical properties $\boldsymbol{e}$ and $\boldsymbol{t}$ (instead of "torsionfree" and "torsion") which fulfill certain requirements, and then the equivalence of (A), (B) and (C) follows as a special case. Therefore we will discuss this more general situation.

Let $\boldsymbol{e}$ and $\boldsymbol{t}$ be two group-theoretical properties which fulfill the following requirements:

(1) 1 is the only group which is at the same time an $\boldsymbol{e}$-group and a $\boldsymbol{t}$-group.

(2) $\boldsymbol{e}$ and $\boldsymbol{t}$ are inherited by subgroups.

(3) $\boldsymbol{t}$ is inherited by factor groups.

(4) The group generated by normal **t**-groups is a **t**-group.

(5) The union of an ascending chain of normal **e**-groups is an **e**-group.

(6) Extensions of **e**-groups resp. **t**-groups by **e**-groups resp. **t**-groups are **e**-groups resp. **t**-groups.

REMARK. Evidently, $\mathbf{e}$ = torsionfree and $\mathbf{t}$ = torsion fulfill (1)–(6). For the sake of brevity, H will always denote an epimorphic image of the group G, and an **e**-**t**-group is a group which is an extension of an **e**-group by a **t**-group. Likewise for a **t**-**e**-group.

THEOREM 1. *The following properties of the group G are equivalent:*

(I) *Each epimorphic image H of G is the extension of an* **e**-*group by a* **t**-*group.*

(II) *If the epimorphic image H of G is not a* **t**-*group, then there exists a normal* **e**-*subgroup $F\neq 1$ of H.*

(III) (a) *If T is a normal* **t**-*subgroup of H, then $H/\mathbf{C}(T)$ is a* **t**-*group.* (b) *If H is not a* **t**-*group, if* 1 *is the only normal* **e**-*subgroup of H and if P is the maximal normal* **t**-*subgroup of H, then $Z(P)=1\neq P$.*

(IV) (a) *If T is a normal* **t**-*subgroup of H, then there exists an epimorphic image H^* of G containing $T^*\trianglelefteq H^*$ such that $T^*\cong T$ and $H/\mathbf{C}(T)\cong H^*/\mathbf{C}(T^*)$ and H^* is a* **t**-*group.* (b) *If H is not an* **e**-**t**-*group, then H is a* **t**-**e**-*group.* (c) *If $A\neq 1$ is an abelian normal* **t**-*subgroup of H and H/A is an* **e**-**t**-*group, then H is an* **e**-**t**-*group.*

(V) (a) *If T is a normal* **t**-*subgroup of H, then $H/\mathbf{C}(T)$ is a* **t**-*group.* (b) *If H is not an* **e**-**t**-*group, then H is a* **t**-**e**-*group.* (c) *If $A\neq 1$ is an abelian normal* **t**-*subgroup of H and H/A is an* **e**-**t**-*group, then H is an* **e**-**t**-*group.*

PROOF. (I)⇒(II). Clearly, this implication holds even without any restrictions on **e** and **t**.

(II)⇒(I). By conditions (5) and (6) there exists a maximal normal **e**-subgroup F in the epimorphic image H of G and 1 is the only normal **e**-subgroup of H/F. Hence by (II) H/F is a **t**-group, so that (I) holds. (Observe that here **t** can be any group-theoretical property and **e** has only to fulfill (5) and (6).)

(I)⇒(III). Let T be any normal **t**-subgroup of an epimorphic image H of G. By (I) there exists a normal **e**-subgroup $N\trianglelefteq H$ with **t**-factorgroup H/N. By condition (1) $T\cap N=1$ and hence $N\leqq \mathbf{C}(T)$. Since **t** fulfills (3), $H/\mathbf{C}(T)$ is a **t**-group and hence (III.a) holds. (III.b) follows by default.

(III)⇒(II). Deny. Hence there exists an epimorphic image H of G which is not a **t**-group and 1 is the only normal **e**-subgroup of H. Let P be the maximal normal **t**-subgroup of H. By condition (4) P

is also a normal $\boldsymbol{t}$-subgroup of H. Thus by (III.b) $Z(P)=1\neq P$ and $1=P\cap \mathbf{C}(P)<P<H$. By (III.a) $H/\mathbf{C}(P)$ is a $\boldsymbol{t}$-group. By condition (6), 1 is the only normal $\boldsymbol{t}$-subgroup of H/P and $H/P\neq 1$ is not a $\boldsymbol{t}$-group. Hence by (III.b) there exists a normal $\mathbf{e}$-subgroup $1\neq W/P \trianglelefteq H/P$. Consider $W\cap\mathbf{C}(P)\cong W\cap \mathbf{C}(P)/P\cap\mathbf{C}(P)\cong P(W\cap\mathbf{C}(P))/P \leqq W/P$.

Hence, by condition (2), $W\cap \mathbf{C}(P)$ is a normal $\mathbf{e}$-subgroup of H and since 1 is the only normal $\mathbf{e}$-subgroup of H, $W\cap\mathbf{C}(P)=1$. Therefore $W=W/W\cap\mathbf{C}(P)\cong W\mathbf{C}(P)/\mathbf{C}(P)\leqq H/\mathbf{C}(P)$, where $H/\mathbf{C}(P)$ is a $\boldsymbol{t}$-group. By condition (2), W is a $\boldsymbol{t}$-group. Hence by condition (3) W/P is a $\boldsymbol{t}$-group. Since now W/P is at the same time an $\mathbf{e}$-group and a $\boldsymbol{t}$-group, by condition (1), $W/P=1$, a contradiction proving (II). (Note: here we have used conditions (1)–(4) and (6).)

(II)$\Rightarrow$(IV). Let T be a normal $\boldsymbol{t}$-subgroup of the epimorphic image H of G. The set of all normal subgroups X of H with $X\cap T=1$ is not empty. By Zorn's Lemma there exists a maximal normal subgroup $M\triangleleft H$ with $M\cap T=1$.

Assume $H^*=H/M$ is not a $\boldsymbol{t}$-group. Hence by (II) there exists a normal $\mathbf{e}$-subgroup $1\neq S/M\trianglelefteq H/M$. Since $M<S$, the maximality of M and condition (2) imply $1\neq S\cap T$ is a normal $\boldsymbol{t}$-subgroup of H. Consider $S/M\geqq M(S\cap T)/M\cong S\cap T/M\cap S\cap T=S\cap T$. Since S/M is an $\mathbf{e}$-group, by condition (2), $S\cap T$ is an $\mathbf{e}$-group. But $S\cap T\neq 1$ is also a $\boldsymbol{t}$-group, contradicting condition (1). Hence H/M is a $\boldsymbol{t}$-group.

Now consider $T^*=TM/M\cong T/M\cap T=T$. Let $\mathbf{C}(T^*)=C/M$. For any x in C we get $x\circ MT\leqq M$ and hence $x\circ T\leqq M$. Since also $x\circ T\leqq T$, $x\circ T=1$. Therefore $C\leqq\mathbf{C}(T)$. If x is in $\mathbf{C}(T)$ then $x\circ MT\leqq M$, so that $\mathbf{C}(T)\leqq C$. Thus $C=\mathbf{C}(T)$. Consider $H^*/\mathbf{C}(T^*)\cong(H/M)/(C/M)$ $\cong H/C=H/\mathbf{C}(T)$. Now from condition (3) (IV.a) follows. Since (II)$\Rightarrow$(I), (IV.b) and (IV.c) are trivially true.

(IV)$\Rightarrow$(V). Clear.

(V)$\Rightarrow$(I). Deny. Hence there exists an epimorphic image H of G which is not an $\mathbf{e}$-$\boldsymbol{t}$-group. Let T be the maximal normal $\boldsymbol{t}$-subgroup of H. By condition (4), T is a $\boldsymbol{t}$-group. Since by (V.b) there exists a normal $\boldsymbol{t}$-subgroup $S\trianglelefteq H$ such that H/S is an $\mathbf{e}$-group, from $S\leqq T$ and conditions (2) and (3) we obtain $S=T$, i.e. H/T is an $\mathbf{e}$-group.

If $Z(T)=1=T\cap\mathbf{C}(T)$, we get $\mathbf{C}(T)\cong T\mathbf{C}(T)/T\trianglelefteq H/T$ and hence $\mathbf{C}(T)$ is a normal $\mathbf{e}$-subgroup of H. By (V.a), $H/\mathbf{C}(T)$ is a $\boldsymbol{t}$-group and therefore H is an $\mathbf{e}$-$\boldsymbol{t}$-group, a contradiction proving $Z(T)=T\cap\mathbf{C}(T)\neq 1$. Again consider $\mathbf{C}(T)/Z(T)\cong T\mathbf{C}(T)/T\trianglelefteq H/T$. Since $\mathbf{e}$ fulfills (2), $\mathbf{C}(T)/Z(T)$ is a normal $\mathbf{e}$-subgroup of $H/Z(T)$ and $(H/Z(T))/(\mathbf{C}(T)/Z(T))\cong H/\mathbf{C}(T)$. Hence, by (V.a), $H/\mathbf{C}(T)$ is a $\boldsymbol{t}$-group and hence $H/Z(T)$ is an $\mathbf{e}$-$\boldsymbol{t}$-group. Since $\boldsymbol{t}$ fulfills (2), $Z(T)\neq 1$

is a normal **t**-subgroup of H. Hence by (V.c), H is an **e**-**t**-group, the desired contradiction and so (I) holds.

Lemma 1. *If $N \trianglelefteq G$, $G \circ N \leqq Z(G)$ and $Z(G)^n = 1$ (for some natural number n), then $N^n \leqq Z(G)$ and N is of exponent n^2.*

Proof. Let $x \in N$ and $g \in G$. From $x \circ g \in N \circ G \leqq Z(G)$ and $Z(G)^n = 1$ we deduce $1 = (x \circ g)^n = x^n \circ g$, i.e. $N^n \leqq \mathbf{C}(g)$. Thus $N^{n^2} \leqq Z(G)^n = 1$.

Proposition 1. *If $Z(G)^n = 1$ (for some natural number n), then $Z_c^{n^c} = 1$ for all natural numbers c.*

Proof. Induce on c.

An immediate consequence of Proposition 1 is the well-known

Proposition 2. *The following properties of the finitely generated nilpotent group G are equivalent:*

(I) *G is finite.*

(II) *$Z(G)$ is finite.*

(III) *$Z(G)$ is a torsion group.*

Another obvious consequence of Proposition 1 is the following

Proposition 3. *If $G = Z_\omega = \bigcup_{k=0}^{\infty} Z_k$ and if $Z(G)^n = 1$ (for some natural number n), then G is an n-group. If $G = Z_k$ (k finite) then $G^{n^k} = 1$.*

Theorem. 2. *If G is hyperabelian and if each abelian normal torsion subgroup of any epimorphic image H of G has finite exponent, then the following properties of G are equivalent.*

(I) *Each epimorphic image H of G induces a torsion group of automorphisms in each of its normal torsion subgroups.*

(II) *Each epimorphic image H of G contains a torsionfree normal subgroup $F \trianglelefteq H$ with torsion factorgroup H/F.*

Proof. (I)$\Rightarrow$(II). Since all hypotheses are inherited by the factorgroups of G, it suffices to prove (II) for G. By factoring out a maximal normal torsionfree subgroup of G we may assume that 1 is the only normal torsionfree subgroup of G. Let M be the maximal normal torsion subgroup of G. Assume $M \neq G$. By the hypercommutativity of $G/M \neq 1$ there exists a torsionfree abelian normal subgroup $1 \neq T/M \trianglelefteq G/M$. Clearly, T is not a torsion group. Let $C = \mathbf{C}(M)$. Hence by (I) G/C is a torsion group, so that $T/T \cap C$ is a torsion group and $S = C \cap T$ is not a torsion group. From $Z(M) = M \cap C < S$ we obtain $S/Z(M) \cong MS/M \leqq T/M$, so that $S' \leqq Z(M) \leqq M$. Hence $S \circ S' = 1$ and S is nilpotent of class $\leqq 2$. Since the torsion subgroup of $Z(S)$ is normal in G, by hypothesis there exists a natural number n

such that $Z(S)^n$ is a torsionfree normal subgroup of G. Hence $Z(S)^n = 1$, so that by Proposition 3 S is a torsion group, a contradiction. Hence $M = G$ and (II) holds.

(II)$\Rightarrow$(I). Again it suffices to prove (I) for G. Apply Theorem 1. Since a noetherian, solvable torsion group is finite, Theorem 2 implies

THEOREM 3 (K. A. Hirsch). *If G is noetherian and solvable, then G contains a torsionfree normal subgroup of finite index.*

REMARK. For the sake of simplicity we have assumed in Theorem 2 that G is hyperabelian. Scrutinizing the proof we see that "hyperabelian" can be replaced by "If 1 is the only normal torsion subgroup of the epimorphic image H of G, then there exists an abelian normal subgroup $A \neq 1$ of H".

Finally we prove

THEOREM 4. *If G is a hypercentral group whose epimorphic images H induce torsion groups of automorphisms in their normal torsion subgroups and if abelian normal torsion subgroups of H have finite exponent, then G is isomorphic to a subgroup of a direct product of a hypercentral torsion group T^* and a hypercentral torsionfree group F^*.*

PROOF. By Theorem 2 there exists a torsionfree normal subgroup F with hypercentral torsion factorgroup $T^* = G/F$. By Baer [1, Theorem 1, p. 193] the torsion elements of G form a characteristic subgroup $T \trianglelefteq G$. Hence $F^* = G/T$ is torsionfree and hypercentral. Consider the direct product $T^* \otimes F^*$ and the mapping $g \rightarrow (gT^*, gF^*) \in T^* \otimes F^*$ for $g \in G$. Evidently, this mapping is an injection of G into $T^* \otimes F^*$.

Since noetherian nilpotent groups fulfill the conditions of Theorem 4, we obtain

THEOREM 5 (K. A. Hirsch). *Noetherian nilpotent groups are isomorphic to a subgroup of a direct product of a finite nilpotent group with a torsionfree noetherian nilpotent group.*

REFERENCES

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