A NOTE ON Z-MAPPINGS AND WZ-MAPPINGS

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1. **Introduction.** In [2] Isiwata introduced the notion of WZ-mappings which he shows to be an extension of the notion of Z-mappings introduced by Frolik [1]. The purpose of this paper is to show that every Z-mapping defined on the space X is closed if and only if X is normal. Necessary and sufficient conditions are also found so that every WZ-mapping is a Z-mapping.

Throughout this paper, topological spaces are assumed to be completely regular T_1-spaces and mappings are continuous functions. If ϕ is a mapping taking X into Y, then Φ will denote the Stone extension of ϕ taking βX into βY.

The mapping $\phi: X \to Y$ is a WZ-mapping if $\mathrm{cl}_{\beta X}\phi^{-1}(y) = \Phi^{-1}(y)$ for each y in the image of ϕ.

The mapping $\phi: X \to Y$ is a Z-mapping if the images of zero-sets are closed.

The space X has property Z if every closed set H is completely separated from every zero-set disjoint from H.

2. **Description of a mapping.** If A is a closed subset of the space X, then let ϕ_A denote the natural function taking X onto $Y = X/A$. Let C denote the subcollection of R^Y to which f belongs if and only if $f \circ \phi_A$ is in $C(X)$. Let the topology on Y be the topology induced by C, i.e., the smallest topology such that each element of C is continuous. Since X is completely regular, Y is Hausdorf; hence, it follows from [**3**, Theorem 3.7] that Y is completely regular. Also, by [**3**, Theorem 3.8], ϕ_A is continuous. The referee has pointed out that the topology on Y is the finest completely regular topology such that ϕ_A is continuous, X has property Z if and only if ϕ_A is a quotient map for each zero-set A in X, and X is normal if and only if ϕ_A is a quotient map for each closed set A in X.

LEMMA. *ϕ_A is always a WZ-mapping.*

PROOF. It need only be shown that $\mathrm{cl}_{\beta X}(A) = \Phi^{-1}(\phi(A))$. Let $x \in \beta X - \mathrm{cl}_{\beta X}(A)$. Since x is not in $\mathrm{cl}_{\beta X}(A)$, there are disjoint zero-set-neighborhoods H and K, such that $x \in H$ and $\mathrm{cl}_{\beta X}(A) \subset K$. By the

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construction of ϕ_A, $\phi_A(H\cap X)$ and $\phi_A(K\cap X)$ are disjoint zero-sets in $Y=X/A$; and so, $\mathrm{cl}_{\beta Y}(\phi_A(H\cap X))$ and $\mathrm{cl}_{\beta Y}(\phi_A(K\cap X))$ are disjoint. It follows that $\phi_A(A)$ does not belong to $\mathrm{cl}_{\beta Y}(\phi_A(H\cap X))$. Since $x\in\mathrm{cl}_{\beta X}(H\cap X)$, $\Phi_A(x)\in\mathrm{cl}_{\beta Y}(\phi_A(H\cap X))$; therefore, $\Phi_A(x)\neq\phi_A(A)$.

3. Results.

THEOREM 1. *X has property Z if and only if every WZ-mapping is a Z-mapping.*

PROOF. First, suppose that X has property Z and that ϕ is a WZ-mapping from X to Y. Let Z be a zero-set in X and y a point of $\phi(X)-\phi(Z)$. Then $\phi^{-1}(y)$ is a closed subset of X. Since X has property Z, there is a zero-set A containing $\phi^{-1}(y)$ that is disjoint from Z. By [3, Theorem 6.5], $\mathrm{cl}_{\beta X}(A)$ is disjoint from $\mathrm{cl}_{\beta X}(Z)$. Since $\Phi^{-1}(y) =\mathrm{cl}_{\beta X}(\phi^{-1}(y))$, $\Phi^{-1}(y)$ is a subset of $\mathrm{cl}_{\beta X}A$; and so, y is not in $\Phi(\mathrm{cl}_{\beta X}Z)$ which is a closed set in βY. It follows that $\phi(Z)$ is closed in $\phi(X)$.

Now, suppose that X does not have property Z; that is, suppose that H is a closed subset of X and K is a zero-set disjoint from H such that H and K are not completely separated. By the lemma, ϕ_H is a WZ-mapping. To see that ϕ_H is not a Z-mapping, observe that $\phi(H)$ is a limit point of $\phi(K)$.

THEOREM 2. *The space X is normal if and only if every Z-mapping is closed.*

PROOF. Suppose that X is not normal.

Case I. X has property Z. Let H and K denote disjoint closed sets that are not completely separated. Consider $\phi_H: X\rightarrow Y=X/H$. ϕ_H is not closed since $\phi_H(H)$ is a limit point of $\phi_H(K)$. However, by the lemma, ϕ_H is a WZ-mapping and hence a Z-mapping by Theorem 1.

Case II. If X does not have property Z, then there are a closed set H and a zero-set K disjoint from H such that H and K are not completely separated. $\phi_K: X\rightarrow X/K$ is not a closed mapping since $\phi_K(K)$ is a limit point of $\phi_K(H)$. However, since any two disjoint zero-sets in X are completely separated, it follows that ϕ_K is a Z-mapping.

Now, in [2] it is shown that a Z-mapping is a WZ-mapping and that if X is normal, then every WZ-mapping defined on X is closed. Thus, if X is normal then every Z-mapping defined on X is closed.

It should be noted that property Z does not imply normality since any countably compact space has property Z. In particular, if Ω denotes the space consisting of the first uncountable segment of the ordinal numbers together with its endpoint ω_1, then $(\Omega\times\Omega)-(\omega_1, \omega_1)$ is a countably compact space that is not normal.

Recall that X is pseudocompact provided that every real-valued mapping defined on X is bounded.

THEOREM 3. *The pseudocompact space X is countably compact if and only if X has property Z.*

PROOF. It is obvious that if X is countably compact it has property Z. Suppose that X is pseudocompact and has property Z but X is not countably compact. Then there is a set $H=\{P_1, P_2, \cdots\}$, $P_i \neq P_j$ for $i \neq j$, such that H has no limit point. Let $\{U_1, U_2, \cdots\}$ be a collection of mutually exclusive open sets such that $P_i \in U_i$ for each i. For each n, let f_n be a mapping taking X into $[0, 1]$ such that $f_n(P_n)=1/n$ and $X-U_n \subset f_n^{-1}(0)$. Let $f=\sup [f_1, f_2, \cdots]$. f is continuous and $f^{-1}(0)$ does not intersect H. By hypothesis, there is a mapping $h: X \rightarrow [0, 1]$ such that $H \subset h^{-1}(0)$ and $f^{-1}(0) \subset h^{-1}(1)$. Let $g=f+h$. Then $1/g$ is a mapping that is unbounded which is a contradiction from which the theorem follows.

From Theorems 1 and 3 the following theorem is obtained.

THEOREM 4. *The pseudocompact space X is countably compact if and only if every WZ-mapping defined on X is a Z-mapping.*

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