HOMOGENEOUS ALMOST TANGENT STRUCTURES

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1. Introduction. Let J be a field of linear operators acting on the complexified tangent spaces of a differentiable manifold V_{2n} , of dimension $2n$, and which satisfies a relation of the form $J^2 = \lambda^2 I$ where λ is a complex constant and I is the identity operator. In the case $\lambda \neq 0$ the manifold has an almost product structure which in the case $\lambda = i$ reduces to an almost complex structure. In the remaining case, $\lambda = 0$, ($\text{rank } J = n$) the manifold has an almost tangent structure [1]. In [2] Frolicher has studied homogeneous almost complex structures and in [3] Legrand has studied homogeneous almost product structures. In the present paper the author extends this treatment to the almost tangent structures.

2. Notation. Let G be a connected Lie group and H a closed subgroup. The quotient space G/H (under the equivalence relation $g_1^{-1}g_2 \in H$) is able to be endowed with the structure of a real analytic manifold V_m with an analytic projection $p: G \rightarrow G/H$. G has with respect to p , a natural structure of a principal fibre bundle with base space V_m and structural group H effective on itself by left translation.

Let us be given an element $\xi \in G$. To the point $x = pg$ (where $g \in G$) of V_m , let us correspond the point

$$K_\xi x = p(\xi g).$$

$K_\xi x$ is independent of the choice of g in $p^{-1}(x)$. G thus appears as a transitive group of analytic transformations of V_m . We will say that V_m , furnished with the structure defined by the group of operators G , is a homogeneous Lie space.

Let the differential of K_ξ at x be denoted by $K'_\xi(x)$. Then $K'_\xi(x)$ is a linear mapping of the tangent vector space at x onto the tangent vector space at $K_\xi x$. In particular at the point $x_0 = pe$ (where e is the identity of G) and for $h \in H$ we have $K'_h(x_0)$ is a linear mapping of T_{x_0} into itself. We may thus represent H by a group $\tilde{H}$ of linear transformations of T_{x_0} . We shall briefly denote $K'_\xi(x_0)$ by K'_ξ .

If $p': T_\theta \rightarrow T_{p\theta}$ denotes the linear mapping of the tangent vector spaces of G onto the tangent vector spaces of V_m we will have for any $u \in T_e$

$$K'_h p' u = p' (\text{adj } h) u$$

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where $(\text{adj } h)$ is the element corresponding to $h \in H$ under the adjoint representation of G .

The extension of the mapping $K'_\xi(x)$ to the tensors defined on T_x will be represented by the same symbol. In particular, for the case of a tensor J of type $(1, 1)$ defined on T_{x_0} we will have

$$K'_\xi(J) = K'_{\xi^{-1}} \circ J \circ K'_\xi.$$

We will assume in the sequel that V_m is of dimension $m = 2n$ for some n .

3. Homogeneous almost tangent structures. Let us consider on a homogeneous Lie space $V_{2n} = G/H$ a field of real or complex tensors; let t_x be the tensor of the field at the point $x = pg$. A necessary and sufficient condition that the field be invariant under G is that

$$(1) \quad t_x = K'_g(t_{x_0})$$

for an arbitrary $g \in p^{-1}(x_0)$, that is, be sent into itself by each element of the group G . Clearly, t_{x_0} is invariant under $\tilde{H}$. From a tensor defined at the point x_0 and invariant under $\tilde{H}$, we may use the formula (1) to define on V_m a field of tensors invariant under G .

In particular let us suppose that there exists at the point x_0 a tensor (F_i^j) , invariant under $\tilde{H}$ and of rank n such that

$$(2) \quad F_i^j F_j^k = 0.$$

One deduces from this by the formula (1) a field of tensors invariant under G and which satisfies the relation (2) at each point of V_{2n} . It thus determines an almost tangent structure on V_{2n} . We will then say that V_{2n} is furnished with a homogeneous almost tangent structure.

If J is a $(1, 1)$ tensor defined on T_{x_0} then it will be invariant under H if and only if

$$K'_h(J) = J$$

that is

$$K'_{h^{-1}} \circ J \circ K'_h = J$$

or

$$J \circ K'_h = K'_h \circ J \quad \text{for any } h \in H.$$

$T_{x_0}^c$ being the complexified vector space of T_{x_0} , a homogeneous almost tangent structure is able to be defined by the data on $T_{x_0}^c$ of a linear operator J , of rank n , satisfying the following conditions:

$$(3) \quad J^2 = 0$$

$$(4) \quad J \circ K'_h = K'_h \circ J \quad \text{for any } h \in H.$$

(3) is the translation of (2) in terms of operators and (4) expresses the invariance of J under $\tilde{H}$.

Let us denote by $\mathbf{G}$ the tangent vector space to G at the point e , furnished with its Lie algebra structure; by $\mathbf{G}^c$ its complexification; and by $\mathbf{H}$ and $\mathbf{H}^c$ the analogous spaces relative to the subgroup H of G . Let us choose in $\mathbf{G}^c$ a subspace M supplementary to $\mathbf{H}^c$, that is such that $\mathbf{G}^c = \mathbf{H}^c \oplus M$. Since any $\mathbf{R}$ -linear map between real vector spaces extends in a unique way to a $\mathbf{C}$ -linear map of the complexified spaces, the mapping $p': \mathbf{G} \rightarrow T_{x_0}$ is able to be extended to $\mathbf{G}^c$. By restriction to M it determines an isomorphism of M onto $T_{x_0}^c$, for which we denote the inverse isomorphism by q' .

To any linear operator J defined on $T_{x_0}^c$ one is able to associate the linear operator $I: \mathbf{G}^c \rightarrow \mathbf{G}^c$ defined by setting $Iu = q'Jp'u$ for any $u \in \mathbf{G}^c$.

If J is of rank n on $T_{x_0}^c$, then $J(T_{x_0}^c)$ is of dimension n and since q' is an isomorphism of $T_{x_0}^c$ onto M then $I(\mathbf{G}^c)$ is an n -dimensional subspace of M . Thus I is a linear operator of rank n , zero on $\mathbf{H}^c$, and is such that $I(\mathbf{G}^c) \cap \mathbf{H}^c = \mathbf{O}$. We note that $p'q'$ is the identity on $J(T_{x_0}^c)$ and $q'p'$ is the identity on M .

Let $h \in H$; one then has for any $u \in \mathbf{G}^c$

$$\begin{aligned} p'(\text{adj } h)Iu &= K'_h p'Iu \\ &= K'_h p'q'Jp'u \\ &= K'_h Jp'u \end{aligned}$$

and

$$\begin{aligned} p'I(\text{adj } h)u &= p'q'Jp'(\text{adj } h)u \\ &= Jp'(\text{adj } h)u \\ &= JK'_h p'u. \end{aligned}$$

Hence

$$(5) \quad p'((\text{adj } h)I - I(\text{adj } h))u = (K'_h J - JK'_h)p'u.$$

Suppose now that J satisfies (3) and (4); then for any $u \in \mathbf{G}^c$ the operator I associated to J is such that:

$$I^2u = q'Jp'q'Jp'u = q'J^2p'u = 0$$

and according to (5) such that:

$$p'((\text{adj } h)I - I(\text{adj } h))u = 0$$

for any $h \in H$. One is then able to write

$$(6) \quad I^2u = 0$$

$$(7) \quad (\text{adj } h)Iu \equiv I(\text{adj } h)u \quad (\text{modulo } H^c)$$

for any $u \in G^c$ and $h \in H$.

Conversely, let I be a linear operator of G^c into itself such that $I(G^c) \cap H^c = O$ where I is zero on H^c , of rank n , and satisfies (6) and (7). Then $I(G^c)$ is an n -dimensional subspace of M where M is supplementary to H^c in G^c . Let q' be, as earlier, the isomorphism of $T_{x_0}^c$ onto M . We define a linear operator J of $T_{x_0}^c$ into itself by setting $Jv = p'Iq'v$ for any $v \in T_{x_0}^c$. Thus J is of rank n . It also follows that

$$J^2v = p'Iq'p'Iq'v = p'I^2q'v = 0$$

for any $v \in T_{x_0}^c$.

Let us put $u = u_H + u_M$ where $u_H \in H^c$, $u_M \in M$, for any $u \in G^c$. We then have $q'v = u_M$ whenever $p'u = v$. Now,

$$\begin{aligned} K'_h Jv &= K'_h p'Iq'v \\ &= p'(\text{adj } h)Iq'v \\ &= p'(\text{adj } h)Iu \end{aligned}$$

where $p'u = v$ since $Iu = I(u_H + u_M) = Iu_M = Iq'v$. In addition we have

$$\begin{aligned} JK'_h v &= p'Iq'K'_h p'u \quad \text{where } p'u = v \\ &= p'Iq'p'(\text{adj } h)u \\ &= p'Iq'p'(w_H + w_M) \quad \text{where } (\text{adj } h)u = w \\ &= p'Iq'p'w_M \\ &= p'Iw_M \\ &= p'I(w_H + w_M) \\ &= p'I(\text{adj } h)u. \end{aligned}$$

Hence for any $v \in T_{x_0}^c$ and any $h \in H$ we have

$$\begin{aligned} (K'_h J - JK'_h)v &= p'((\text{adj } h)I - I(\text{adj } h))u \\ &= 0 \end{aligned}$$

where $p'u = v$.

We have thus proven

THEOREM 1. *In order that a homogeneous Lie space $V_{2n} = G/H$ be able to be endowed with a homogeneous almost tangent structure it is*

necessary and sufficient that there exists on G^c a linear operator I satisfying the following conditions:

- (a) I is zero on H^c ,
- (b) $\text{rank } I = n$,
- (c) $I(G^c) \cap H^c = 0$,
- (d) $I^2 u = 0$ for any $u \in G^c$,
- (e) $(\text{adj } h)Iu \equiv I(\text{adj } h)u \pmod{H^c}$ for any $u \in G^c$ and $h \in H$.

If (7) is satisfied for any $h \in H$, it will in particular be satisfied for the elements $h = \exp(tz)$ of the one parameter subgroup generated by an element z of H . By passage to the Lie algebras one deduces from this that the condition (7) leads to

$$(8) \quad [z, Iu] \equiv I[z, u] \pmod{H^c}$$

for any $u \in G^c$ and $z \in H$. I being a linear operator on G^c , this relation extends to the elements $z \in H^c$.

Conversely, if (8) is satisfied, the condition (7) is satisfied for the elements $h \in H$ such that $(\text{adj } h)$ belongs to a one parameter subgroup generated by an element of the Lie algebra of the group $(\text{adj } H)$. If this group is connected, (7) is then satisfied for arbitrary h .

THEOREM 2. *In order that a homogeneous Lie space $V_{2n} = G/H$ be able to be endowed with a homogeneous almost tangent structure, it is necessary that there exists on G^c a linear operator I satisfying the following conditions:*

- (a) I is zero on H^c ,
- (b) $\text{rank } I = n$,
- (c) $I(G^c) \cap H^c = 0$,
- (d) $I^2 u = 0$ for any $u \in G^c$,
- (e) $[Iu, u'_H] \equiv I[u, u'_H] \pmod{H^c}$ for any $u \in G^c$ and $u'_H \in H^c$.

These conditions are sufficient if $(\text{adj } H)$ is connected.

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