A BOUND FOR SOLUTIONS OF VOLTERRA-STIELTJES INTEGRAL EQUATIONS

R. H. MARTIN, JR.

W. A. Coppel [1, p. 58] uses a "measure" μ defined on square matrices by $\mu[A] = \lim_{h \rightarrow +0} (|1 + hA| - 1)/h$ in order to obtain a bound for solutions to the linear differential equation $y' = Ay$. This paper extends this result to a Volterra-Stieltjes integral equation

$$y(x) = 1 + \int_x^z dF \cdot y$$

where F is of bounded variation on each bounded subinterval and the integral is the Cauchy right integral. Initial value problems of this type have been investigated by J. S. MacNerney and others. See [4] for references.

Notations and definitions are similar to those in [3] and we indicate them here. Let S denote the set of real numbers and R a complete normed algebra over the real field with norm $|\cdot|$ and unity 1 such that $|1| = 1$. The algebra of real numbers will be considered as a subalgebra of R . A function B from $S \times S$ into R is said to be order-additive (respectively, order-multiplicative) if (x, z) is in $S \times S$ and $x \leq y \leq z$ or $x \geq y \geq z$, then $B(x, y) + B(y, z) = B(x, z)$ (respectively, $B(x, y) \cdot B(y, z) = B(x, z)$). Let OA^+ be the set of all order-additive functions from $S \times S$ into the nonnegative real numbers and OM^+ be the set of all order-multiplicative functions from $S \times S$ into the real numbers not less than one. Let OA be the set of all order-additive functions V from $S \times S$ into R for which there is an α in OA^+ such that $|V| \leq \alpha$, and let OM be the set of all order-multiplicative functions W from $S \times S$ into R for which there is a ν in OM^+ such that $|W - 1| \leq \nu - 1$. The set OB shall consist of all functions F from S into R such that dF is in OA —where $dF(x, y)$ is defined to be $F(y) - F(x)$. Note that F is in OB if and only if F is of bounded variation on each bounded subinterval of S .

If (a, b) is in $S \times S$, then $(t_p)_0^n$ is a subdivision of (a, b) if $t_0 = a$, $t_n = b$ and $t_{p-1} \leq t_p$ or $t_{p-1} \geq t_p$ according to whether $a \leq b$ or $a > b$. If h is a function from $S \times S$ to R , then ${}_a \sum^b h \sim \sum_t h = \sum_1^n h(t_{p-1}, t_p)$ and ${}_a \prod^b h \sim \prod_t h = \prod_1^n h(t_{p-1}, t_p)$ and the limit is in the sense of refinements of subdivisions. For each α in OA^+ , V in OA and (a, b) in $S \times S$ let $\varepsilon^+[\alpha](a, b) = {}_a \prod^b [1 + \alpha]$ and $\varepsilon[V](a, b) = {}_a \prod^b [1 + V]$. Mac-

Received by the editors March 28, 1969.

Nerney [3, Theorems 2.2, 3.3, and 4.2] shows that ε^+ is a bijection from OA^+ into OM^+ , ε is a bijection from OA to OM and for each P in R and a in S there is a unique U in OB such that

$$U(z) = P + (R) \int_z^a V \cdot U$$

for each z in S . Furthermore, $U(z) = \varepsilon[V](z, a) \cdot P$. $(R) \int_z^a V \cdot U$ is approximated by $\sum_1^n V(t_{p-1}, t_p) \cdot U(t_p)$ where $(t_p)_n$ is a subdivision of (z, a) .

Using the techniques of Coppel [1, p. 3] one can show that for each A in R ,

$$\mu[A] = \lim_{h \rightarrow +0} \frac{|1 + hA| - 1}{h}$$

exists. Furthermore, (see [1, p. 41] and references cited there) if A and B are in R , then $\mu[rA] = r\mu[A]$ for each positive real number r , $|\mu[A]| \leq |A|$, $\mu[A+B] \leq \mu[A] + \mu[B]$ and $|\mu[A] - \mu[B]| \leq |A - B|$.

DEFINITION. For each A in R let $\exp(A) = \sum_0^\infty A^p/(p!)$.

LEMMA 1. If A is in R , then

- (i) $\mu[A] = \lim_{h \rightarrow +0} (|\exp(hA)| - 1)/h$,
 - (ii) $|\exp(A)| \leq \exp(\mu[A])$, and
 - (iii) $1 + h\mu[A] \leq |1 + hA| \leq 1 + h\mu[A] + 2(\exp(h|A|) - 1 - h|A|)$
- for each $h \geq 0$. In particular, if $2h|A| \leq 1$,

$$|1 + hA| \leq 1 + h\mu[A] + 4h^2|A|^2.$$

INDICATION OF PROOF. If $2h|A| \leq 1$, then $|\exp(hA) - 1 - hA| \leq 2h^2|A|^2$ and (i) follows immediately. By (i), for each $\epsilon > 0$ there is an integer $n \geq 1$ such that $|(|\exp(A/n)| - 1)n - \mu[A]| < \epsilon$. Hence,

$$\begin{aligned} |\exp(A)| &\leq |\exp(A/n)|^n = [1 + (|\exp(A/n)| - 1)n/n]^n \\ &\leq [1 + (\mu[A] + \epsilon)/n]^n \leq \exp(\mu[A]) + \epsilon \cdot \exp(2|\mu[A]| + \epsilon) \end{aligned}$$

and (ii) follows. If $0 < k < h$, then $(|1 + kA| - 1)/k \leq (|1 + hA| - 1)/h$ which shows that $1 + h\mu[A] \leq |1 + hA|$. Furthermore, if $h \geq 0$, then $\mu[hA] = h\mu[A] \leq h|A|$ so

$$\begin{aligned} |1 + hA| &\leq |\exp(hA)| + \sum_2^\infty h^p|A|^p/(p!) \\ &\leq \exp(\mu[hA]) + (\exp(h|A|) - 1 - h|A|) \end{aligned}$$

and (iii) follows.

LEMMA 2. Suppose that V is in OA , α is in OA^+ such that $|V| \leq \alpha$, and (a, b) is in $S \times S$. Then whenever t is a subdivision of (a, b) and s is a refinement of t , $\sum_t \mu[V] \leq \sum_s \mu[V] \leq \alpha(a, b)$, so that $\sum^b \mu[V]$ exists.

LEMMA 3. Let V be in OA , (a, b) in $S \times S$ and α in OA^+ such that $|V| \leq \alpha$, then

(i) if there is a subdivision $(t_p)_0^n$ of (a, b) such that $\alpha(t_{p-1}, t_p) \leq \delta \leq 1/2$ for each $1 \leq p \leq n$, then

$$\left| \sum_t (|1 + V| - 1) - \sum_t \mu[V] \right| \leq 4\delta\alpha(a, b),$$

(ii) for each $\delta > 0$ there is a subdivision $(u_p)_0^n$ of (a, b) such that if $u_{p-1} < x \leq y < u_p$ or $u_{p-1} > x \geq y > u_p$, then $\alpha(x, y) \leq \delta$, and

(iii) for each x in S and each $\epsilon > 0$, there is a $\delta > 0$ such that if $x < y \leq x + \delta$, then $\alpha(y, x + \delta) \leq \epsilon$, and if $x > y \geq x - \delta$, then $\alpha(x - \delta, y) \leq \epsilon$.

INDICATION OF PROOF. By using part (iii) of Lemma 1,

$$\left| \sum_t (|1 + V| - 1) - \sum_t \mu[V] \right| \leq \sum_1^n 4 |V(t_{p-1}, t_p)|^2$$

and (i) follows. Both (ii) and (iii) are true since α can be uniformly approximated by step functions on bounded subintervals of S .

THEOREM 1. Suppose that V is in OA and (a, b) is in $S \times S$, then $\sum^b (|1 + V| - 1)$ exists.

INDICATION OF PROOF. Suppose that $a < b$, ϵ is a positive number less than $1/2$, and α is in OA^+ such that $|V| \leq \alpha$. By (ii) and (iii) of Lemma 3, there is a subdivision $(u_p)_0^n$ of (a, b) and sequences $(x_p)_1^n$ and $(y_p)_1^n$ so that if $1 \leq p \leq n$,

1. $u_{p-1} < x_p < y_p < u_p$ and $\alpha(x_p, y_p) \leq \epsilon$,
2. if $u_{p-1} < z \leq x_p$, then $\alpha(z, x_p) \leq \epsilon/n$, and
3. if $y_p \leq z < u_p$, then $\alpha(y_p, z) \leq \epsilon/n$.

By Lemma 2, for each $1 \leq p \leq n$, there is a subdivision s^p of (x_p, y_p) such that if t is a refinement of s^p , then $|\sum_t \mu[V] - \sum_{s^p} \mu[V]| \leq \epsilon/n$. Let $(s(q))_0^m$ be a subdivision of (a, b) whose range equals the union of the range of u and the range of s^p for each $1 \leq p \leq n$. We see that

$$\begin{aligned} \sum_s (|1 + V| - 1) &= \sum_1^n \left\{ |1 + V(u_{p-1}, x_p)| - 1 \right. \\ &\quad \left. + \sum_{s^p} (|1 + V| - 1) + |1 + V(y_p, u_p)| - 1 \right\}. \end{aligned}$$

Suppose that $(t_q)_0^1$ is a refinement of s . We can estimate $\left| \sum_t (|1+V|-1) - \sum_s (|1+V|-1) \right|$ by comparing the differences in each of the intervals $[u_{p-1}, x_p]$, $[x_p, y_p]$ and $[y_p, u_p]$ for $1 \leq p \leq n$. Let $1 \leq p \leq n$ and suppose that q_1 and q_2 are integers such that $u_{p-1} = t(q_1) < t(q_1+1)$ and $t(q_2) = x_p$. In the interval $[u_{p-1}, x_p]$ we have the estimate

$$\begin{aligned} & \left| \sum_{q_1+1}^{q_2} [|1+V(t(k-1), t(k))| - 1] - (|1+V(u_{p-1}, x_p)| - 1) \right| \\ & \leq |1+V(u_{p-1}, t(q_1+1))| - |1+V(u_{p-1}, x_p)| + \alpha(t(q_1+1), x_p) \\ & \leq 2\alpha(t(q_1+1), x_p) \\ & \leq 2\epsilon/n. \end{aligned}$$

A similar estimate holds in the interval $[y_p, u_p]$. Now suppose q_1 and q_2 are integers such that $t(q_1) = x_p$ and $t(q_2) = y_p$ and let r be the subdivision of (x_p, y_p) such that $r(k) = t(q_1+k)$ for each $0 \leq k \leq q_2 - q_1$. By using part (i) of Lemma 3, we have

$$\begin{aligned} & \left| \sum_r (|1+V| - 1) - \sum_{s^p} (|1+V| - 1) \right| \\ & \leq \left| \sum_r (|1+V| - 1) - \sum_r \mu[V] \right| + \left| \sum_r \mu[V] - \sum_{s^p} \mu[V] \right| \\ & \quad + \left| \sum_{s^p} \mu[V] - \sum_{s^p} (|1+V| - 1) \right| \\ & \leq 4\epsilon\alpha(x_p, y_p) + \epsilon/n + 4\epsilon\alpha(x_p, y_p) \\ & \leq 8\epsilon\alpha(u_{p-1}, u_p) + \epsilon/n. \end{aligned}$$

Combining the above estimates shows that

$$\left| \sum_t (|1+V| - 1) - \sum_s (|1+V| - 1) \right| \leq (5 + 8\alpha(a, b)) \cdot \epsilon$$

and the assertion of the theorem follows.

If V is in OA the function $\gamma[V]$ from $S \times S$ into R defined by $\gamma[V](x, y) = \sum^y (|1+V|-1)$ is clearly order-additive. The inequality $||1+V(x, y)|-1| \leq |V(x, y)|$ shows that $\gamma[V]$ is in OA and if α is in OA^+ such that $|V| \leq \alpha$, then $|\gamma[V]| \leq \alpha$.

Let OAI denote the set of all V in OA such that for each x in S , $1+V(x, x+)$, $1+V(x+, x)$, $1+V(x-, x)$ and $1+V(x, x-)$ have multiplicative inverses in R . J. V. Herod [2] shows that $\mathcal{E}[V](a, b)$ has a multiplicative inverse in R for each (a, b) in $S \times S$ if and only

if V is in OAI. Furthermore, there is a bijection G from OAI into OAI so that each of the following holds.

- (1) $G(V)(a, b) = -\delta \sum^a (1+V)^{-1} V$ for each (a, b) in $S \times S$,
- (2) $G(V)(a, b) = -V(b, a)$ for each (a, b) in $S \times S$ if and only if ${}_a \sum^b |V^2| = 0$ for each (a, b) in $S \times S$, and
- (3) $\varepsilon[G(V)](a, b) \cdot \varepsilon[V](b, a) = 1$ for each (a, b) in $S \times S$.

LEMMA 4. *If V is in OA, (a, b) is in $S \times S$ and ϵ is a positive number, there is a subdivision u of (a, b) such that whenever t is a refinement of u , $\sum_t |\gamma[V] - (|1+V| - 1)| \leq \epsilon$.*

THEOREM 2. *Suppose that V is in OA and α is in OA^+ so that $|V| \leq \alpha$. Then for each (a, b) in $S \times S$*

- (i) $|\varepsilon[V](a, b)| \leq \varepsilon[\gamma[V]](a, b) \leq \varepsilon^+[\alpha](a, b)$.

If, in addition, V is in OAI and β is in OA^+ so that $|\gamma[G(V)]| \leq \beta$, then

- (ii) $\varepsilon[V](a, b) \geq \{\varepsilon[\gamma[G(V)]](b, a)\}^{-1} \geq \{\varepsilon^+[\beta](b, a)\}^{-1}$.

REMARKS. Compare with Coppel [1, p. 58]. Also, it can be shown that, with the supposition of Theorem 2, $|\gamma[G(V)]| \leq G(\alpha)$.

INDICATION OF PROOF. If ϵ is a positive number there is a subdivision t of (a, b) such that

- (1) $|\varepsilon[V](a, b)| \leq \prod_t (1+V) + \epsilon$,
- (2) $\varepsilon[\gamma[V]](a, b) \leq \prod_t (1+\gamma[V]) + \epsilon$,

and

- (3) $\sum_t |\gamma[V] - (|1+V| - 1)| \leq \epsilon$.

Hence,

$$|\varepsilon[V](a, b)| \leq \prod_1^n (1 + \gamma[V](t_{p-1}, t_p) + s_p) + \epsilon$$

where $s_p = |1 + V(t_{p-1}, t_p)| - 1 - \gamma[V](t_{p-1}, t_p)$. Since

$$\begin{aligned} & \prod_1^n (1 + \gamma[V](t_{p-1}, t_p) + s_p) - \prod_1^n (1 + \gamma[V](t_{p-1}, t_p)) \\ &= \sum_1^n \left[\prod_1^{p-1} (1 + \gamma[V](t_{q-1}, t_q)) \right] \cdot s_p \cdot \left[\prod_{p+1}^n (1 + \gamma[V](t_{q-1}, t_q) + s_p) \right] \end{aligned}$$

and $\sum_1^n |s_p| \leq \epsilon$, it follows that

$$|\varepsilon[V](a, b)| \leq \varepsilon[\gamma[V]](a, b) + \epsilon \cdot (2 + \exp(2\alpha(a, b) + \epsilon)).$$

Part (i) now follows. If V is in OAI, $\varepsilon[G(V)](b, a) \cdot \varepsilon[V](a, b) = 1$ and it follows that $|\varepsilon[V](a, b)| \geq |\varepsilon[G(V)](b, a)|^{-1}$. The assertion of the theorem follows by applying part (i) to $G(V)$.

EXAMPLE. Let a be in S , f be a continuous function from S into R and let $F(z) = \int_z^a f(x)dx$ for each z in S . Part (iii) of Lemma 1 shows that $(|1 + hf| - 1)/h$ tends to $\mu[f]$ uniformly on bounded subintervals of S . If (a, b) is in $S \times S$ and t is a subdivision of (a, b) , then estimates of the type

$$\begin{aligned} \gamma[dF](a, b) &\sim \sum_1^n \left(\left| 1 + \int_{t_p}^{t_{p-1}} f(x)dx \right| - 1 \right) \\ &\sim \sum_1^n \frac{|1 + (t_{p-1} - t_p)f(t_{p-1})| - 1}{(t_{p-1} - t_p)} (t_{p-1} - t_p) \end{aligned}$$

show that

$$\gamma[dF](a, b) = - \int_b^a \mu[-f(x)]dx \quad \text{if } a \leq b,$$

and

$$\gamma[dF](a, b) = \int_b^a \mu[f(x)]dx \quad \text{if } a \geq b.$$

Hence, if P is in R and U is in OB such that $U(z) = \varepsilon[dF](z, a) \cdot P$ for each z in S , part (i) of Theorem 2 shows that

$$\begin{aligned} |U(z)| &\leq |P| \exp \left(\int_a^z \mu[f(x)]dx \right) \quad \text{if } z \geq a, \\ &\leq |P| \exp \left(- \int_a^z \mu[-f(x)]dx \right) \quad \text{if } z \leq a. \end{aligned}$$

Since dF is in OAI and $\sum^y_x |dF^2| = 0$ for each (x, y) in $S \times S$, $G(V)(x, y) = -dF(y, x)$ and part (ii) of Theorem 2 shows that

$$\begin{aligned} |U(z)| &\geq |P| \exp \left(- \int_a^z \mu[-f(x)]dx \right) \quad \text{if } z \geq a, \\ &\geq |P| \exp \left(\int_a^z \mu[f(x)]dx \right) \quad \text{if } z \leq a. \end{aligned}$$

Consequently, Theorem 2 contains Theorem 3 of [1, p. 58].

REFERENCES

1. W. A. Coppel, *Stability and asymptotic behavior of differential equations*, Heath, Boston, 1965.
2. J. V. Herod, *Multiplicative inverses of solutions for Volterra-Stieltjes integral equations*, Proc. Amer. Math. Soc. **22** (1969), 650–656.
3. J. S. MacNerney, *Integral equations and semigroups*, Illinois J. Math. **7** (1963), 148–173.
4. ———, *A linear initial value problem*, Bull. Amer. Math. Soc. **69** (1963), 314–329.

GEORGIA INSTITUTE OF TECHNOLOGY