A NOTE ON RAYS AT THE IDENTITY OPERATOR

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Let $\mathfrak{B}(\mathfrak{X})$ be the algebra of all bounded linear automorphisms of a Banach space $\mathfrak{X}$ and let S be the unit ball of $\mathfrak{B}(\mathfrak{X})$. If I is the identity operator and $A \neq 0$, the ray generated by A (through I) is the set $\{I + tA : t \geq 0\}$. This ray is said to be tangent to S at I if

$$(*) \quad \lim_{t \rightarrow 0+} t^{-1}(\|I + tA\| - 1) = 0.$$

Bohenblust and Karlin [1] raised the question of existence of rays tangent to S at I from the radical. Lumer and Phillips [3] settled the conjecture in the negative by constructing a quasi-nilpotent operator on Hilbert space for which $(*)$ obtains. For this purpose the negative of the standard Volterra operator on the L_2 space of the interval $[0, 1]$ will serve as well as their example. By a modification of their example we give an example of a ray from the radical which is not only tangent to S at I but is actually in contact with the surface of the ball on an interval of the ray.

Let N be any nonzero quasi-nilpotent in $\mathfrak{B}(\mathfrak{X})$ which satisfies $(*)$. Any operator A which satisfies $(*)$ is dissipative and so generates a semigroup of contractions; $\|e^{tA}\| \leq 1$ for $t \geq 0$. If we set $Q = e^N - I$ we see that Q is an analytic function of N so is itself a quasi-nilpotent by the spectral mapping theorem. Now $\|I + tQ\| = \|e^{tN}\| = 1$ so the ray from Q is in contact with S for $t = 0$ and $t = 1$. From the convexity of S it follows that $\|I + tQ\| \leq 1$ for $0 \leq t \leq 1$ and from the fact that the norm must exceed the spectral radius $\|I + tQ\| \geq 1$. Thus for $0 \leq t \leq 1$ the ray is on the surface of S .

We have thus far begged the question that $Q \neq 0$. For the Volterra operator N on $L_2[0, 1]$ we have $\|N\| = 1$ and it follows at once from examination of the exponential series that $e^{-N} = I$ is not possible.

If N is an arbitrary quasi-nilpotent on a Hilbert space with $e^N = I$ it follows at once that e^{tN} is a bounded group so that Nagy's theorem that a bounded representation of the integers is similar to a unitary representation applies. The spectral mapping theorem then finishes this case.

For the setting of an arbitrary Banach space we can argue as follows. Let $T = e^{t_0 N}$. If $e^N = I$ then we have a uniform bound $\|T^n\| \leq B$

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for $-\infty < n < \infty$. Again from the spectral mapping theorem we have $\sigma(T) = \{1\}$ so that T is a quasi-nilpotent perturbation of the identity. Then by a classical result of Gel'fand's [2, p. 128] T is the identity operator. Thus $(t_0)^{-1}(I - e^{t_0 N}) = 0$ on one hand and tends to N as $t_0 \rightarrow 0$ on the other.

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