ABSTRACT DIFFERENTIAL ALGEBRA AND THE ANALYTIC CASE. II

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Let R be the rational number-field, $K = R\langle u_1, \dots, u_m \rangle$ a finite ordinary differential extension-field of R . Let $u_i \mapsto c_{ij}$ under an arbitrary abstract-field isomorphism σ of K onto a field K_1 . Let the field of formal power-series $K_1((z))$ be converted into a differential field by placing $(\sum a_j z^j / j!)' = \sum a_j z^{j-1} / (j-1)!$. Then the assignment $u_i \mapsto \bar{u}_i = \sum c_{ij} z^j / j!$ defines a differential-field isomorphism of K onto $R\langle \bar{u}_1, \dots, \bar{u}_m \rangle$.

This is easily proved (see [1, p. 160]). We may note that on $R\{u_1, \dots, u_m\}$ the mapping σ is given by $H(u) \mapsto H(\bar{u})|_{z=0}$ (for $H \in R\{U\}$). Thus if $H(u) \mapsto H(\bar{u})|_{z=0}$ is an isomorphism, then $H(u) \mapsto H(\bar{u})$ is an isomorphism. The converse is not true: thus suppose $m=1$ and $u=u_1$ is transcendental over R ; then $H(u) \mapsto H(\bar{u} - \bar{u}(0))$ still yields an isomorphism, but $H(u) \mapsto H(\bar{u} - \bar{u}(0))|_{z=0}$ does not.

As, however, in the proof of a corollary in [1, p. 162] we tacitly made use of this converse, a correction is necessary. This corollary reads:

Let K, K_1 be finite extensions of the rational number-field R , $K \subseteq K_1$, and let $\tau: K \rightarrow K^$ be an isomorphism of K with a field of meromorphic functions K^* . Then τ can be extended to an isomorphism $\tau: K_1 \rightarrow K_1^*$ of K_1 with a field of meromorphic functions K_1^* .*

In the definition of a field of meromorphic functions K^* , an open, connected region A intervenes. If A_1 is an open, connected subregion of A , the mapping $f \mapsto f|_{A_1}$ (for $f \in K^*$) yields an isomorphism of K^* . It is understood in the above corollary that K^* may be replaced by any isomorphic copy obtained in this way.

Let $K = R\langle u_1, \dots, u_m \rangle$, $\tau: u_i \mapsto \bar{u}_i$, and let A be the region associated with $K^* = R\langle \bar{u}_1, \dots, \bar{u}_m \rangle$. The functions $\bar{u}_1, \dots, \bar{u}_m$ need not be holomorphic, but writing them as quotients of holomorphic functions, letting P be a point of A where the denominators do not vanish, and suitably restricting A about P , we may assume that they are holomorphic. We may also assume that P is given by $z=0$. Now consider the mapping $H(u) \mapsto H(\bar{u})|_{z=0}$. This is always a homomorphism, but may fail to be an isomorphism. However, if $H(\bar{u}) \neq 0$, then $H(\bar{u})$ has only a finite number of zeros in a small circle about P ; we propose to select a new P , avoiding these zeros. Since there are

only an enumerable number of H in $R\{U\}$, we can choose a new P so that $H(u) \neq 0$ implies $H(\bar{u})|_P \neq 0$. Thus, although the converse spoken about is not true, we operate very much as though it were.

Let $K_1 = K\langle v_1, \dots, v_n \rangle$. Clearly we need consider only the case $n=1$, $K_1 = K\langle v \rangle$. In case v is transcendental over K_1 , we take $\bar{v} = \sum d_j z^j / j!$ with $d_0, d_1, \dots$, algebraically independent over $R(\dots, c_{ij}, \dots)$ and $\sum d_j z^j / j!$ convergent, where $\bar{u}_i = \sum c_{ij} z^j / j!$ and $H(u) \mapsto H(\bar{u})|_{z=0}$ is an isomorphism after the preparations of the last paragraph. Then

$$R(\dots, c_{ij}, \dots, d_0, d_1, \dots) \simeq R(\dots, u_{ij}, \dots, v_0, v_1, \dots)$$

and $u_i \mapsto \bar{u}_i$, $v \mapsto \bar{v}$ defines an isomorphism of $K\langle v \rangle$ onto a field of meromorphic functions; this isomorphism extends τ .

Now consider the case that v is algebraic over K . By the previous case, we may assume that some u_i are transcendental over R , by adjoining an extra u if necessary, hence, by the theorem of the primitive element, may assume that $u_1, \dots, u_{m-1}$ are algebraically independent and that u_m and v are primitive elements of K and K_1 respectively over $R\langle u_1, \dots, u_{m-1} \rangle$. After any such preparation, if we lose the assumption that $H(u) \mapsto H(\bar{u})|_{z=0}$ is an isomorphism, we can always regain it. Now let $v, v_1, \dots, v_{r-1}$ be algebraically independent (in the algebraic sense) over $R\langle u_1, \dots, u_{m-1} \rangle$ and v_r algebraic over $R\langle u_1, \dots, u_{m-1} \rangle\langle v, \dots, v_{r-1} \rangle$. Let $\bar{v} = \sum d_j z^j / j!$ be constructed so that $\bar{v}$ is convergent and $R(\{c_{ij}\}, \dots, \{c_{m-1j}\}, \{d_j\}) \simeq R(\{u_{ij}\}, \dots, \{u_{m-1j}\}, \{v_j\})$ —see the proof of the Embedding Theorem [1, p. 160]. As in that proof, $\bar{v}$ is the unique solution of a differential equation $V_r - d_r = T(z; V_0 - d_0, \dots, V_{r-1} - d_{r-1})$ with specified initial conditions. Hence in this construction we can always replace $\{d_j\}$ by a conjugate $\{d_j^*\} / R(\dots, c_{ij}, \dots)$ ($i=1, \dots, m-1$) to get an equally good $\sum d_j^* z^j / j!$ (i.e., $\sum d_j^* z^j / j!$ will also be convergent, as it is the unique solution of the differential equation $V_r - d_r^* = T(z; V_0 - d_0^*, \dots, V_{r-1} - d_{r-1}^*)$ with specified initial conditions).

Now we have an isomorphic mapping of $R\langle u_1, \dots, u_{m-1}, v \rangle$ onto a field of meromorphic functions $R\langle \bar{u}_1, \dots, \bar{u}_{m-1}, \bar{v} \rangle$; and contracting A , we may suppose $R\{\bar{u}_1, \dots, \bar{u}_{m-1}, \bar{v}\}$ consists of holomorphic functions in A . This isomorphism agrees with τ on $u_1, \dots, u_{m-1}$. However, it may send u_m into a $u_m^* \neq \bar{u}_m$; in fact, u_m^* may not even be holomorphic. (The possibility that $u_m^* \neq \bar{u}_m$ was not taken into account in [1, p. 162].) Changing the center $z=0$, we may assume u_m^* is holomorphic, too; $u_m^* = \sum c_{mj}^* z^j / j!$. Moreover we may assume

$$R(\{u_{1j}\}, \dots, \{u_{m-1j}\}, \{u_{mj}\}, \{v_j\}) \\ \simeq R(\{c_{1j}\}, \dots, \{c_{m-1j}\}, \{c_{mj}^*\}, \{d_j\}).$$

Now

$$R(\{c_{1j}\}, \dots, \{c_{m-1j}\}, \{c_{mj}^*\}) \simeq R(\{c_{1j}\}, \dots, \{c_{m-1j}\}, \{c_{mj}\})$$

and this isomorphism can be extended to

$$R(\{c_{1j}\}, \dots, \{c_{m-1j}\}, \{c_{mj}^*\}, \{d_j\}) \\ \simeq R(\{c_{1j}\}, \dots, \{c_{m-1j}\}, \{c_{mj}\}, \{d_j^*\}).$$

These d_j^* define a $\bar{v}^* = \sum d_j^* z^j / j!$, which can replace $\bar{v}$. I.e., we have an isomorphism $R\langle u_1, \dots, u_m, v \rangle \simeq R\langle \bar{u}_1, \dots, \bar{u}_m, \bar{v}^* \rangle$. This isomorphism extends τ and the proof is complete.

Above we were assuming that K, K_1 were ordinary differential fields. However, similar considerations hold for partial differential fields.

REFERENCE

1. A. Seidenberg, *Abstract differential algebra and the analytic case*, Proc. Amer. Math. Soc. 9 (1958), 159-164.

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