FINITE GENERATION OF GROTHENDIECK RINGS RELATIVE TO CYCLIC SUBGROUPS¹

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Let Ω be a field of characteristic p ($p \neq 0$), G a finite group, and consider finitely generated left ΩG -modules (" G -modules" for short). The isomorphism classes of G -modules, relative to direct sum, form a free $\mathbb{Z}$ -module $a(G)$ called the *representation ring* of G . Let $\mathcal{U}$ be some family of subgroups of G . An exact sequence of ΩG -modules

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

is called $\mathcal{U}$ -split if its restriction to each $H \in \mathcal{U}$ is split. Let $i(G, \mathcal{U})$ be the ideal of $a(G)$ generated by all expressions $[M] - [L] - [N]$ arising from $\mathcal{U}$ -split G -exact sequences. Then

$$a(G, \mathcal{U}) = a(G)/i(G, \mathcal{U})$$

is the *relative Grothendieck ring*, and is a commutative ring with unity. Finally, we use the notation $M|N$ to indicate that M is a direct summand of N ; M_H denotes the restriction of M , and V^G an induced module.

Let us say that a G -module M is $(G, \mathcal{U})$ -projective if

$$M \Big| \sum_{H \in \mathcal{U}}^{\oplus} (M_H)^G.$$

(This generalizes the usual notion when $\mathcal{U}$ consists of a single subgroup of G .) The $(G, \mathcal{U})$ -projective modules generate an ideal $k(G, \mathcal{U})$ in $a(G)$, and the map

$$\kappa: k(G, \mathcal{U}) \rightarrow a(G, \mathcal{U})$$

given by $[M] \rightarrow [M]$ is the *Cartan map*. As shown by Dress [2], κ is monic and its cokernel is a p -torsion group.

The present authors have investigated $a(G, H)$ extensively, and have shown [4], [7] that:

(i) If H is a normal cyclic p -subgroup of G , then $a(G, H)$ has a finite free $\mathbb{Z}$ -basis.

(ii) If H is any cyclic subgroup of a p -group G , then $a(G, H)$ has a finite free $\mathbb{Z}$ -basis (and indeed, $a(G, H) \cong a(H)$).

We cannot expect that $a(G, H)$ will always have a *finite* free

Received by the editors May 26, 1969.

¹ This research was partially supported by the National Science Foundation.

Z -basis; in fact, when $H=G$ the relative ring $a(G, H)$ is just $a(G)$, and this is finitely generated if and only if G has a cyclic p -Sylow subgroup, by [3]. On the other hand, all of the available evidence [4]–[7] indicates that $a(G, H)$ is always Z -free, but this is a difficult open question.

The purpose of this article is to establish the comforting theorem:

If H is cyclic, then $a(G, H)$ is a finitely generated Z -module.

This will appear as a special case of the Main Theorem given below.

We shall also sketch our calculations showing that $a(G, H)$ has a finite free Z -basis when $G=A_4$ or S_4 , and $|H|=2$, where $p=2$.

MAIN THEOREM. *The relative Grothendieck ring $a(G, \mathbf{U})$ is finitely generated as Z -module if and only if each $H \in \mathbf{U}$ has a cyclic p -Sylow subgroup.*

PROOF. In one direction the result is straightforward: Assume $a(G, \mathbf{U})$ finitely generated, and let $H \in \mathbf{U}$. There is an epimorphism $a(G, \mathbf{U}) \rightarrow a(G, H)$, whence $a(G, H)$ is finitely generated. Since κ embeds $k(G, H)$ into $a(G, H)$, also $k(G, H)$ is finitely generated. This implies at once (as in [3], say) that H has a cyclic p -Sylow subgroup.

Conversely, assume that each $H \in \mathbf{U}$ has a cyclic p -Sylow subgroup. Replacing each H by one of its p -Sylow subgroups does not affect $\mathbf{U}$ -splitting or $(G, \mathbf{U})$ -projectivity, so $a(G, \mathbf{U})$ and $k(G, \mathbf{U})$ are unchanged. We may thus assume hereafter that $\mathbf{U} = \{H_1, \dots, H_d\}$ is some collection of cyclic p -subgroups of G , and we must show that $a(G, \mathbf{U})$ is finitely generated.

Let $F'_1, \dots, F'_s$ be a full set of irreducible ΩG -modules. They can all be realized in some finite subfield Ω_0 of Ω . In fact, let F'_i afford the (ordinary) character ϕ_i , and let Ω_0 be obtained from the prime field $GF(p)$ of Ω by adjoining all of the character values $\{\phi_i(g) : g \in G, 1 \leq i \leq s\}$. Then (see [1, Theorem 70.23]) each F'_i is realizable in Ω_0 , that is, there exist $\Omega_0 G$ -modules F_i with $F'_i = \Omega F_i$, $1 \leq i \leq s$.

Now let $H \in \mathbf{U}$ be cyclic with generator a of order h , and set $\alpha = a - 1 \in \Omega H$, so $\alpha^h = 0$. Let F be an $\Omega_0 H$ -submodule of the ΩH -module M , and put $n = (F : \Omega_0)$. We now prove by induction on n that there exists an $\Omega_0 H$ -submodule T of M such that

$$(T : \Omega_0) \leq hn, \quad F \subset T \subset M, \quad \Omega T \mid M.$$

Let $n \geq 1$ and assume the result known if $(F : \Omega_0) < n$. Since H is a p -group, there exists a nonzero H -trivial element $x \in F$. Let $x \in \alpha^m M$ with m maximal, and write $x = \alpha^m y$, $y \in M$. Note that $\alpha x = 0$ since x

is H -trivial. Now set $Y = \Omega_0 H \cdot y$; we claim that $\Omega Y = \Omega H \cdot y$ is an H -direct summand of M , and for this it suffices to prove that

$$\alpha^r M \cap \Omega Y = \alpha^r \Omega Y, \quad 0 \leq r \leq h-1.$$

Now each $u \in \Omega Y$ is expressible as $u = \sum_0^m a_i \cdot \alpha^i y$, $a_i \in \Omega$. Let $u \in \alpha^r M$, and let a_j be the first nonzero coefficient, so

$$u = a_j \cdot \alpha^j y + \cdots + a_m \cdot \alpha^m y \in \alpha^r M.$$

Multiply by α^{m-j} , getting $a_j x \in \alpha^{r+m-j} M$, whence (since m was maximal) $j \geq r$. Then $u \in \alpha^j \Omega Y \subset \alpha^r \Omega Y$, as desired, and we have shown that $\Omega Y \mid M$.

Next observe that $(F+Y)/Y$ is an $\Omega_0 H$ -submodule of $M/\Omega Y$ of Ω_0 -dimension $\leq n-1$, so by the induction hypothesis there exists an $\Omega_0 H$ -submodule T/Y of $M/\Omega Y$ such that

$$(T/Y: \Omega_0) \leq h(n-1), \quad \frac{F+Y}{Y} \subset \frac{T}{Y} \subset \frac{M}{\Omega Y}, \quad \frac{\Omega T}{\Omega Y} \mid \frac{M}{\Omega Y}.$$

Here T is itself an $\Omega_0 H$ -submodule of M , and $Y \subset F+Y \subset T \subset M$. Now write

$$M = \Omega T + L, \quad L \cap \Omega T = \Omega Y, \quad L = \Omega H\text{-submodule of } M;$$

this is possible since $\Omega T/\Omega Y$ is a direct summand of $M/\Omega Y$. But now $\Omega Y \mid M$, $\Omega Y \subset L$, so $\Omega Y \mid L$; hence $L = \Omega Y \oplus W$, say. Then $M = \Omega T \oplus W$, proving that $\Omega T \mid M$. Finally,

$$(T: \Omega_0) = (T/Y: \Omega_0) + (Y: \Omega_0) \leq h(n-1) + h = hn.$$

This completes the induction proof.

We apply the above procedure to the irreducible $\Omega_0 G$ -modules $F_1, \dots, F_s$ and an arbitrary ΩG -module M . Namely, if $F_r \subset M$ then for each $H \in \mathcal{U}$ we can find an $\Omega_0 H$ -submodule T of M satisfying

$$F_r \subset T \subset M, \quad \Omega T \mid M_H, \quad (T: \Omega_0) \text{ bounded},$$

with the bound independent of M . Hence there exists a positive constant c , depending only on the group G , with the following property:

If F is any irreducible $\Omega_0 G$ -module contained in an ΩG -module M , then for each i , $1 \leq i \leq d$, there exists an $\Omega_0 H_i$ -submodule N_i of M satisfying the conditions

$$F \subset N_i \subset M, \quad (N_i: \Omega_0) \leq c, \quad \Omega N_i \mid M_{H_i}.$$

We shall refer to the system

$$[F; N_1, \dots, N_d; \alpha_1, \dots, \alpha_d],$$

where each α_i is the $\Omega_0 H_i$ -embedding of F in N_i , as a *system* derived from M . Obviously each nonzero M gives rise to one or more such systems.

Let M' be another ΩG -module containing the same F , and let the system $[F; N'_1, \dots, N'_d; \alpha'_1, \dots, \alpha'_d]$ be derived from M' . We call this system *equivalent* to the preceding one if there exist $\Omega_0 H_i$ -isomorphisms $\nu_i: N_i \cong N'_i$ ($1 \leq i \leq d$) such that $\nu_i \alpha_i = \alpha'_i$, $1 \leq i \leq d$. Now each N_i is a vector space over the finite field Ω_0 , and the dimensions of all $\{N_i\}$ are uniformly bounded. Hence the number of equivalence classes of systems which begin with any one of $F_1, \dots, F_s$, must be finite.

For each σ , $1 \leq \sigma \leq s$, choose a finite set of ΩG -modules $\{M'_\sigma: 1 \leq r \leq u_\sigma\}$ whose derived systems

$$[F_\sigma; N_{1,\sigma}^r, \dots, N_{d,\sigma}^r; \alpha_{1,\sigma}^r, \dots, \alpha_{d,\sigma}^r]$$

represent all equivalence classes of systems which begin with F_σ . We claim that the elements

$$(1) \quad \{[M'_\sigma] - [M'_\sigma/\Omega F_\sigma]: 1 \leq r \leq u_\sigma, 1 \leq \sigma \leq s\}$$

generate the additive group $a(G, \mathbf{U})$. Indeed, let M be any nonzero ΩG -module, and let us show by induction on $(M: \Omega)$ that $[M]$ is expressible as a $\mathbb{Z}$ -linear combination of the above elements. Pick any irreducible ΩG -submodule $\Omega F_\sigma \subset M$, and form a derived system $[F_\sigma; N_1, \dots, N_d; \alpha_1, \dots, \alpha_d]$, so that

$$F_\sigma \subset N_i \subset M, \Omega N_i \mid M_{H_i}, \quad 1 \leq i \leq d.$$

Choose r so that this system is equivalent to the one derived above from M'_σ . Then for each i , $1 \leq i \leq d$, the diagram

$$\begin{array}{ccc} \Omega F_\sigma & \xrightarrow{\quad} & M \\ \downarrow & \nearrow \rho_i & \nearrow \tau_i \\ M'_\sigma & & \end{array}$$

in which the horizontal and vertical arrows are ΩG -inclusions, can always be completed by ΩH_i -homomorphisms ρ_i, τ_i to a commutative diagram. Indeed, we have:

$$\begin{array}{ccc} \Omega F_\sigma & \xrightarrow{\alpha_i} & \Omega N_i \mid M_{H_i} \\ \alpha_{i,\sigma}^r \downarrow & & \\ \Omega N_{i,\sigma}^r & \mid & (M_\sigma^r)_{H_i} \end{array}$$

But there exists an ΩH_i -isomorphism $\nu_i: \Omega N_i \cong \Omega N_{i,\sigma}^r$ with $\nu_i \alpha_i = \alpha_{i,\sigma}^r$. Hence it suffices to choose $\rho_i = \nu_i$ on ΩN_i , $\rho_i = 0$ on a complement of ΩN_i in M_{H_i} ; likewise, let $\tau_i = \nu_i^{-1}$ on $\Omega N_{i,\sigma}^r$, and $\tau_i = 0$ on a complement of $\Omega N_{i,\sigma}^r$ in $(M_\sigma^r)_{H_i}$.

It now follows that if P denotes the ΩG -module defined by the pushout diagram

$$\begin{array}{ccc} \Omega F_\sigma & \rightarrow & M \\ \downarrow & & \downarrow \\ M_\sigma^r & \rightarrow & P, \end{array}$$

then the sequences

$$0 \rightarrow M_\sigma^r \rightarrow P \rightarrow M/\Omega F_\sigma \rightarrow 0, \quad 0 \rightarrow M \rightarrow P \rightarrow M_\sigma^r/\Omega F_\sigma \rightarrow 0$$

are H_i -split for each $H_i \in \mathcal{U}$. Thus in $a(G, \mathcal{U})$ we have the relation

$$[M] = [M_\sigma^r] - [M_\sigma^r/\Omega F_\sigma] + [M/\Omega F_\sigma].$$

The last term is a combination of the elements in (1), by the induction hypothesis. Hence so is $[M]$, and the theorem is proved.

Of course this procedure is rather wasteful, since we obtain far too many generators for $a(G, \mathcal{U})$. However, if it is true that $a(G, \mathcal{U})$ is $\mathbb{Z}$ -free, then since its $\mathbb{Z}$ -rank is that of $k(G, \mathcal{U})$, it can be generated by just as many elements as needed to generate $k(G, \mathcal{U})$, and we can presumably determine the latter quite explicitly.

We now show what happens when $p=2$, $\mathcal{U} = \{H\}$, $|H|=2$, $G=A_4$ or S_4 , to indicate the difficulties that one faces in trying to reduce the number of generators for $a(G, H)$.

Consider first $A_4 = \langle a, c \mid a = (12)(34), c = (123) \rangle$, $H = \langle a \rangle$, Ω any field of characteristic 2 containing ω , a primitive cube root of 1. We claim $a(A_4, H)$ is $\mathbb{Z}$ -free on 4 generators. There are 3 irreducible A_4 -modules, given by

$$F_i: a \rightarrow 1, c \rightarrow \omega^i, \quad i = 0, 1, 2.$$

For $i \neq j$ there is an A_4 -module Y_{ij} with

$$0 \rightarrow F_i \rightarrow Y_{ij} \rightarrow F_j \rightarrow 0 \text{ exact, } (Y_{ij})_H \cong \Omega H, F_i = \alpha \cdot Y_{ij},$$

where $\alpha = a - 1$.

Let M be any A_4 -module containing F_i . If $F_i \not\subseteq \alpha M$, then since $(F_i: \Omega) = 1$ we have $F_i \cap \alpha M = 0 = \alpha F_i$, and so F_i is an H -direct summand of M . Hence when $F_i \subseteq \alpha M$ we have

$$[M] = [F_i] + [M/F_i] \quad \text{in } a(A_4, H).$$

On the other hand if $F_i \subset \alpha M$, then there is a system derived from M of the form $[F_i; N; \alpha_1]$ in which $F_i \subset N \subset M$, $N \mid M_H$, $(N: \Omega) = 2$, $F_i = \alpha N$. But for $j \neq i$, Y_{ij} gives a system equivalent to this one. Hence we obtain in $a(A_4, H)$

$$(2) \quad [M] = [Y_{ij}] - [Y_{ij}/F_i] + [M/F_i],$$

and of course $[Y_{ij}/F_i] = [F_j]$. It follows at once that the 6 elements $[F_0]$, $[F_1]$, $[F_2]$, $[Y_{10}]$, $[Y_{21}]$, $[Y_{02}]$ generate $a(A_4, H)$. Further, (2) yields identities such as

$$(3) \quad [Y_{10}] - [F_0] = [Y_{12}] - [F_2], [Y_{20}] - [F_0] = [Y_{21}] - [F_1]$$

etc. Finally let $C = \langle c \rangle$, $1_C =$ trivial C -module of dimension 1. It turns out that there are a pair of H -split A_4 -exact sequences:

$$0 \rightarrow Y_{01} \rightarrow (1_C)^{A_4} \rightarrow Y_{20} \rightarrow 0,$$

$$0 \rightarrow Y_{02} \rightarrow (1_C)^{A_4} \rightarrow Y_{10} \rightarrow 0,$$

so $[Y_{20}] + [Y_{01}] = [Y_{02}] + [Y_{10}]$. Using (3) we obtain

$$[Y_{10}] = [Y_{20}] - [F_2] + [F_1].$$

Multiplying by $[F_1]$, we get $[Y_{21}]$ in terms of $[Y_{01}]$ and the F 's. This result, together with equations (3), show that $a(A_4, H)$ is spanned by the 4 elements $[F_0]$, $[F_1]$, $[F_2]$ and $[Y_{10}]$. Using the maps $a(A_4, H) \rightarrow a(H)$ and $a(A_4, H) \rightarrow a(A_4, 1)$, it is easily seen that $a(A_4, H)$ is Z -free on these 4 generators. In fact it follows that $\phi: a(A_4, H) \rightarrow a(H) \oplus a(A_4, 1)$ is a monomorphism. This means that the class of any A_4 -module M in $a(A_4, H)$ depends only on the composition factors of M and on the H -isomorphism type of M_H . This remark enables us to resolve any $[M]$ in $a(A_4, H)$ as a combination of the four indicated generators. The ring monomorphism ϕ shows also that $a(A_4, H)$ is without nilpotent elements. Finally, routine calculation shows that $\text{coker } \kappa$ is a $(2, 2)$ -group.

Now let $G = S_4$, and keep all of the above notation. We wish to investigate $a(G, H)$ with H of order 2. If $H = \langle (12) \rangle$, then put $K =$ Klein 4-group, so $K \trianglelefteq G$, $K \cap S_3 = 1$, $G = K \cdot S_3$. By the Excision Theorem [5, 3.2] we obtain $a(G, H) \cong a(S_3, H) = a(S_3)$, the latter since H is a 2-Sylow subgroup of S_3 . Hence $a(G, H)$ is easily computed in this case.

Suppose hereafter that $H = \langle a \rangle$, with $a = (12)(34)$ as above, and set $c = (123)$, $d = (1324)$, $\delta = d - 1$, so $\alpha = \delta^2$, and c, d generate G . There are two irreducible Ω -representations of G :

$$E_0: c \rightarrow 1, d \rightarrow 1, \quad E_1: c \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, d \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $E_1 = F_1^G = F_2^G$, with F_1, F_2 the representations of A_4 defined previously. There exist G -modules X_{01}, X_{10}, X_{11} giving nonsplit extensions

$$0 \rightarrow E_i \rightarrow X_{ij} \rightarrow E_j \rightarrow 0, \quad (i, j) = (0, 1), (1, 0), (1, 1),$$

and such that

$$E_0 = \alpha X_{01}, \quad E_1 = \alpha X_{11}, \quad (E_1 \cap \alpha X_{10}: \Omega) = 1.$$

Indeed we have

$$X_{11} = Y_{12}^G, \quad X_{01} = Y_{01}^G/E_0,$$

where E_0 is embedded in Y_{01}^G via $E_0 \rightarrow F_0^G \subset Y_{01}^G$. The relation

$$[Y_{12}] + [F_0] = [Y_{01}] + [F_2] \quad \text{in } a(A_4, H)$$

gives

$$(4) \quad [X_{11}] + [F_0^G] = [Y_{01}^G] + [E_1] \quad \text{in } a(G, H).$$

Now F_0^G is H -trivial, so $[F_0^G] = 2[E_0]$. Also, $E_0 \subset \alpha \cdot Y_{01}^G$, so Y_{01}^G gives the system $[E_0; N]$, $N \cong \Omega H$. But X_{01} gives an equivalent system, so

$$(5) \quad [X_{01}] - [X_{01}/E_0] = [Y_{01}^G] - [Y_{01}^G/E_0].$$

From (4) and (5) we get

$$(6) \quad [X_{11}] = 2[X_{01}] - 2[E_0] \quad \text{in } a(G, H).$$

As a last preliminary step, we note the existence of a G -module L with both sequences exact:

$$(7) \quad 0 \rightarrow E_0 \rightarrow L \rightarrow X_{11} \rightarrow 0, \quad 0 \rightarrow E_1 \rightarrow L \rightarrow X_{01} \rightarrow 0,$$

and

$$E_0 \not\subset \alpha L, \quad (E_1 \cap \alpha L: \Omega) = 1.$$

We are now ready to prove that $[E_0], [E_1], [X_{01}]$ are a free $\mathbb{Z}$ -basis of $a(G, H)$. Let M be any nonzero G -module. If $E_0 \subset M$, we have

Case i. $E_0 \not\subset \alpha M$. Then $E_0 \cap \alpha M = 0 = \alpha E_0$, so $(E_0)_H \mid M_H$, and $[M] = [E_0] + [M/E_0]$ in $a(G, H)$.

Case ii. $E_0 \subset \alpha M$. Then M yields a system $[E_0; N]$ with $N = \Omega H$, $N|_{M_H}$, $E_0 = \alpha N$. Since X_{01} yields an equivalent system, we obtain $[M] = [X_{01}] - [X_{01}/E_0] + [M/E_0]$, and here $[X_{01}/E_0] = [E_1]$.

On the other hand let $E_1 \subset M$.

Case iii. $E_1 \cap \alpha M = 0$. Then $E_1 \cap \alpha M = \alpha E_1$, so $[M] = [E_1] + [M/E_1]$.

Case iv. $E_1 \subset \alpha M$. Then M yields the system $[E_1; N]$, $E_1 = \alpha N$, $N \cong \Omega H + \Omega H$, equivalent to that gotten from X_{11} . Hence $[M] = [X_{11}] - [X_{11}/E_1] + [M/E_1]$.

Case v. $(E_1 \cap \alpha M : \Omega) = 1$. There is a unique one-dimensional subspace $\Omega e \subset E_1$ such that $\delta e = 0$, where $\delta = d - 1$. But $\delta(E_1 \cap \alpha M) \subset E_1 \cap \alpha M$, whence $\delta(E_1 \cap \alpha M) = 0$, and so $E_1 \cap \alpha M = \Omega e$. This shows that $E_1 \cap \alpha M$ is independent of M .

Now $E_1 \cap \alpha X_{10} = \Omega e$, and we may write $e = \alpha x$, $x \in X_{10}$; choose $z \in X_{10}$ with

$$E_1 = \Omega \cdot \alpha x \oplus \Omega z, \quad X_{10} = \Omega x \oplus \Omega \cdot \alpha x \oplus \Omega z.$$

Also $E_1 \cap \alpha M = \Omega e$, so $e = \alpha m$ for some $m \in M$. The embedding $E_1 \rightarrow M$ carries αx onto αm , and takes z onto some element $n \in M$. Set $N = \Omega m \oplus \Omega \alpha m \oplus \Omega n$; then $N|_{M_H}$ since $N \cap \alpha M = \alpha N$. There is an H -isomorphism $\tau: X_{10} \cong N$ given by $x \rightarrow m$, $\alpha x \rightarrow \alpha m$, $z \rightarrow n$, making the diagram

$$\begin{array}{ccc} E_1 & \subset & N \\ \downarrow & \nearrow \tau & \\ X_{10} & & \end{array}$$

commute. Extend τ to a map of X_{10} into M , and define a map $M \rightarrow X_{10}$ by letting it coincide with τ^{-1} on N , and vanish on an H -complement of N . The Pushout Lemma [5, (3.1)] gives

$$[M] = [X_{10}] - [X_{10}/E_1] + [M/E_1] = [X_{10}] - [E_0] + [M/E_1].$$

These arguments show that $[E_0]$, $[E_1]$, $[X_{10}]$ and $[X_{01}]$ generate $a(G, H)$. By applying Cases (i) and (v) to the module L in (7), we obtain

$$[L] = [E_0] + [X_{11}] = [X_{10}] - [E_0] + [X_{01}].$$

By (6) this gives $[X_{10}] = [X_{01}]$, so indeed the generator $[X_{01}]$ may be omitted. Using the maps $a(G, H) \rightarrow a(G, 1)$ and $a(G, H) \rightarrow a(H)$, it is easily seen that $a(G, H)$ is $\mathbb{Z}$ -free on $[E_0]$, $[E_1]$, $[X_{10}]$. Also, as in the case of A_4 , the class of $[M] \in a(G, H)$ is uniquely determined by M_H

and the S_4 -composition factors of M ; furthermore $a(S_4, H)$ is free of nonzero nilpotent elements.

Unfortunately we have not been able to generalize these methods to establish freeness of $a(G, H)$ in the case where H is *any* (non-normal) cyclic p -group.

REFERENCES

1. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Interscience, New York, 1962.
2. A. Dress, *On relative Grothendieck rings*, Bull. Amer. Math. Soc. **75** (1969), 955–958.
3. D. G. Higman, *Indecomposable representations at characteristic p* , Duke Math. J. **21** (1954), 377–381.
4. T. Y. Lam and I. Reiner, *Relative Grothendieck groups*, J. Algebra **11** (1969), 213–242.
5. ———, *Reduction theorems for relative Grothendieck rings*, Trans. Amer. Math. Soc. **142** (1969), 421–435.
6. ———, *Relative Grothendieck rings*, Bull. Amer. Math. Soc. **75** (1969), 496–498.
7. ———, *Restriction maps on relative Grothendieck rings*, J. Algebra (to appear).

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