ON A THEOREM OF REPRESENTATION OF LATTICES

S. P. WANG[1]

Let G be a connected Lie group, Γ a discrete subgroup of G and μ a fixed right Haar measure of G. μ induces a measure $\bar{\mu}$ over G/Γ, the homogeneous space of left cosets of Γ. Γ is called a *lattice of G* if $\bar{\mu}(G/\Gamma)$ is finite. We denote the set of all representations of Γ in G, i.e., all homomorphisms from Γ into G by $\mathcal{R}(\Gamma, G)$. We shall give $\mathcal{R}(\Gamma, G)$ the compact-open topology, Let H be any closed subgroup of G. We use U^H to mean the *induced unitary representation* of G by the trivial one dimensional unitary representation of H. In this note, we are going to prove the following

PROPOSITION. *Let G be a connected semisimple Lie group without compact factor, Γ a lattice of G and $\{r_n\}$ a sequence of representations of Γ in G. If $\Gamma_n = r_n(\Gamma)$ are discrete for all n and* $\lim_n r_n = 1_\Gamma$, *then the trivial representation of Γ is weakly contained in $\{\oplus U^{\Gamma_n}|\Gamma\}$.*[2]

As a consequence of the preceding proposition and results in [3] and [6], one can establish the following

THEOREM. *Let G be a connected semisimple Lie group with each factor of $\mathbf{R}$-rank $\geqq 2$, Γ a lattice of G and $\{r_n\}$ a sequence of representations of Γ in G. If $r_n(\Gamma)$ are all discrete and* $\lim_n r_n = 1_\Gamma$, *then $r_n(\Gamma)$ are lattices of G for sufficiently large values of n.*

In general the theorem is not true without the restriction on the $\mathbf{R}$-rank of each factor of G. This can be illustrated by the classical counterexample in the case of the upper half-plane: the Hecke groups Γ_c generated by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix},$$

for $c \geqq 2$. They are all isomorphic to Γ_2, but only Γ_2 is a lattice.

Before we give the proofs of the proposition and the theorem, let us verify some lemmas needed later.

LEMMA 1. *Let G be a semisimple Lie group without compact factor, Γ a lattice of G and $\{r_n\}$ a sequence of representations of Γ in G such that $\Gamma_n = r_n(\Gamma)$ are all discrete. If* $\lim_n r_n = 1_\Gamma$, *then we have*

Received by the editors November 9, 1968.

[1] The author would like to thank the referee for helpful suggestions.

[2] For weak containment, see [2].

(i) $\{\Gamma_n\}$ *is uniformly discrete, i.e., there exists a neighborhood* V *of identity* e *in* G *with* $\Gamma_n \cap V = \{e\}$ *for all* n.

(ii) $\Gamma \cap Z(G)$ *is of finite index in* $Z(G)$ *and* $r_n|_{\Gamma \cap Z(G)} = 1_{\Gamma \cap Z(G)}$ *for sufficiently large value of* n, *where* $Z(G)$ *is the center of* G.

PROOF. Since Γ is a lattice, by the density theorem of lattices [1], $Z(G)\Gamma$ is also a lattice. Hence it follows easily that $\Gamma \cap Z(G)$ is of finite index in $Z(G)$. Let $\hat{G}$ be the Lie algebra of G and $\mathrm{Ad}: G \to \mathrm{GL}(\hat{G})$ the adjoint representation of G. By the density theorem [1], $l(\mathrm{Ad}(\Gamma))$, the linear span of $\mathrm{Ad}(\Gamma)$ in $\mathrm{End}(\hat{G})$, equals $l(\mathrm{Ad}(G))$. Since $\lim_n r_n = 1_\Gamma$, $\dim l(\mathrm{Ad}(\Gamma_n)) = \dim l(\mathrm{Ad}(\Gamma)) = \dim l(\mathrm{Ad}(G))$ for sufficiently large n. Therefore $l(\mathrm{Ad}(\Gamma_n)) = l(\mathrm{Ad}(G))$ holds for large n. It follows in particular that the $r_n(\Gamma \cap Z(G))$ are central for large n. As $Z(G)$ is discrete and $\Gamma \cap Z(G)$ finitely generated, $r_n|_{\Gamma \cap Z(G)} = 1_{\Gamma \cap Z(G)}$ for sufficiently large n.

LEMMA 2. *Let* G *be a Lie group,* μ *a Haar measure, and* U, V, W *three* μ*-measurable subsets of* G. *If* $\mu(V-W) + \mu(W-V) < \epsilon$, *then*

$$|\mu(U - V) - \mu(U - W)| < \epsilon.$$

PROOF. $U - V = (U - (V \cup W)) \cup ((W - V) \cap U)$ and $U - W = (U - (V \cup W)) \cup ((V - W) \cap U)$.
Hence

$$|\mu(U - V) - \mu(U - W)| \leqq \mu(V - W) + \mu(W - V) < \epsilon.$$

COROLLARY 3. *Let* U *and* V *be two finite* μ*-measurable open subsets of* G *and* $(\gamma_1^{(m)}, \cdots, \gamma_n^{(m)}) \in G^n$ *such that* $\lim_m(\gamma_1^{(m)}, \cdots, \gamma_n^{(m)}) = (\gamma_1, \cdots, \gamma_n) \in G^n$.
Then

$$\lim_m \mu(U - V\gamma_1^{(m)} - \cdots - V\gamma_n^{(m)}) = \mu(U - V\gamma_1 - \cdots - V\gamma_n).$$

PROOF. Let W be any finite μ-measurable open subset of G. Then the function $f_W(x) = \mu(W - Wx)$, $x \in G$ is continuous. Hence

$$\lim_m \mu(V\gamma_i - V\gamma_i^{(m)}) = 0 \quad \text{and} \quad \lim_m \mu(V\gamma_i^{(m)} - V\gamma_i) = 0,$$

for $1 \leqq i \leqq n$. Then by Lemma 2, the corollary follows immediately.

COROLLARY 4. *Let* U *and* V *be two finite* μ*-measurable open subsets,* Γ *a discrete subgroup of* G *and* $\{r_n\}$ *a sequence of representations of* Γ *in* G *such that* $\lim_n r_n = 1_\Gamma$. *Then* $\lim_n \sup \mu(U - V\Gamma_n) \leqq \mu(U - V\Gamma)$ *where* $\Gamma_n = r_n(\Gamma)$ *for all* n.

PROOF. As Γ is a discrete subgroup of G, Γ is countable. Let $\Gamma=\{\gamma_1, \cdots, \gamma_i, \cdots\}$ and $U_n=U-V\gamma_1-\cdots-V\gamma_n$. Then $\{U_n\}$ is a decreasing sequence of finite μ-measurable subsets of G. Since $\bigcap_n U_n=U-V\Gamma$, $\lim_n \mu(U_n)=\mu(U-V\Gamma)$. Let ϵ be any positive number. There exists a positive integer N such that $0\leqq\mu(U_n)-\mu(U-V\Gamma)<\epsilon$ for all $n\geqq N$. Let $\gamma_i^{(m)}=r_m(\gamma_i)$, $i=1, 2, \cdots$. Clearly $\lim_m (\gamma_1^{(m)}, \cdots, \gamma_n^{(m)})=(\gamma_1, \cdots, \gamma_n)$. Thus by Corollary 3,

$$\lim_m \mu(U - V\gamma_1^{(m)} - \cdots - V\gamma_N^{(m)}) = \mu(U_N) < \mu(U - V\Gamma) + \epsilon.$$

Since $\mu(U-V\Gamma_m)\leqq\mu(-V\gamma_1^{(m)}-\cdots-V\gamma_N^{(m)})$ for all m, $\lim_m \sup \mu(U-V\Gamma_m)\leqq\mu(U-V\Gamma)+\epsilon$. However ϵ is an arbitrary positive number. This implies

$$\limsup_n \mu(U - V\Gamma_n) \leqq \mu(U - V\Gamma).$$

Now we are ready to prove the proposition and the theorem.

PROOF OF THE PROPOSITION. By Lemma 1, there exists an open neighborhood W of e in G such that $W^{-1}W\cap\Gamma_n=\{e\}$ for all n. By constructing it properly, we can find a fundamental domain F of Γ such that $W\subset F$ and the boundary of F has 0 μ-measure. Let V be the interior of F. It is clear that $\mu(V)=\bar{\mu}(G/\Gamma)<\infty$ and $\mu(xV-V\Gamma)=0$ for all $x\in G$. Therefore by Corollary 4, $\lim_n \mu(xV-V\Gamma_n)=0$. Let $\bar{\mu}_n$ be the invariant measure of G/Γ_n induced from μ. Then we have

$$\begin{aligned}\bar{\mu}_n(xV\Gamma_n - V\Gamma_n) &= \bar{\mu}_n((xV - V\Gamma_n)\Gamma_n)\\ &\leqq \mu(xV - V\Gamma_n), \qquad \text{for all } n,\end{aligned}$$

and all $x\in G$. Hence $\lim_n \bar{\mu}_n(xV\Gamma_n-V\Gamma_n)=0$. Since $W^{-1}W\cap\Gamma_n=\{e\}$ and $W\subset V$, $\bar{\mu}_n(V\Gamma_n)\geqq\mu(W)$. Now let $\{\gamma_1, \cdots, \gamma_m\}$ be any finite subset of Γ. We have $\lim_n \bar{\mu}_n(\gamma_i V\Gamma_n-V\Gamma_n)/\bar{\mu}_n(V\Gamma_n)=0$, $1\leqq i\leqq m$. Now denote the characteristic function of $V\Gamma_n$ in G/Γ_n by χ_n, i.e., $\chi_n(g\Gamma)=1$ or 0 according as $g\in V\Gamma_n$ or not, and $f_n=\bar{\mu}_n(V\Gamma_n)^{-1/2}\chi_n$. By the definition of induced representation, f_n is in the Hilbert space of the unitary representation U^{Γ_n}. Due to our construction, $\lim_n\langle\gamma_i f_n, f_n\rangle=1$, $1\leqq i\leqq m$. Therefore the trivial representation of Γ is weakly contained in $\{\oplus U^{\Gamma_n}|\Gamma\}$.

PROOF OF THE THEOREM. We shall prove it first for the case when G has no nontrivial center. Suppose the theorem to be false in this case. Then there is a subsequence (n') of (n) such that $\Gamma_{n'}$ are not lattices of G for all n'. By the proposition, the trivial representation I of Γ is weakly contained in $\{\oplus U^{\Gamma_{n'}}|\Gamma\}$. However Γ has property (T) of [3] and [6]. It follows that I is a subrepresentation of $\oplus U^{\Gamma_{n'}}|\Gamma$. Therefore $I\leqq U^{\Gamma_{n'}}|\Gamma$ for certain n', say n_0'. Then by Lemma

3.4 in [6], $U^{\Gamma_{n'_0}}$ contains the trivial representation of G. By Lemma 8.2 in [4], $\Gamma_{n'_0}$ has to be a lattice of G, which contradicts our choice of (n'). Next we shall reduce the general case to the previous one. Let $Z(G)$ be the center of G. By Lemma 1 (ii), $Z(G)\cap\Gamma$ is a subgroup of finite index in $Z(G)$ and $r_n|_{Z(G)\cap\Gamma}=1_{Z(G)\cap\Gamma}$ for sufficiently large n. Denote $G/Z(G)$, $\Gamma Z(G)/Z(G)$, $\Gamma_n Z(G)/Z(G)$ by $\bar{G}$, $\bar{\Gamma}$ and $\bar{\Gamma}_n$ respectively. Clearly for large n, r_n induce r_n: $\bar{\Gamma}\to\bar{G}$ such that $r_n(\bar{\Gamma})=\bar{\Gamma}_n$. Since $r_{n|Z(G)\cap\Gamma}=1_{Z(G)\cap\Gamma}$ for large n and $Z(G)\cap\Gamma$ is of finite index in $Z(G)$, the $\bar{\Gamma}_n$ are discrete for large n. Therefore $\bar{G}$, $\{\bar{r}_n\}$, and $\bar{\Gamma}_n$ satisfy the conditions of the theorem and $\bar{G}$ has no nontrivial center. By what we have just proved, the $\bar{\Gamma}_n$ are lattices of $\bar{G}$ for large n. Since Γ_n contains $Z(G)\cap\Gamma$, which is of finite index in $Z(G)$, for large n, one concludes readily that the Γ_n are lattices of G for sufficiently large n.

As suggested by the referee, the proposition and theorem can be generalized slightly in the following case. Γ is a finitely generated group, r_0: $\Gamma\to G$ a homomorphism whose image is a lattice, and r_n: $\Gamma_n\to G$ homomorphisms whose images are discrete, and which tend to r_0. It is easy to see that the slightly more general form can be obtained from the following lemma and our results.

LEMMA 5. *Let* G *be a connected Lie group*, Γ *a finitely generated group*, r_0: $\Gamma\to G$ *a homomorphism whose image is discrete, and* r_n: $\Gamma\to G$ *homomorphisms whose images are uniformly discrete and which tend to* r_0. *Then there exist* $r'_n\in R(r_0(\Gamma), G)$ *such that* $r_n=r'_n\cdot r_0$ for large n and $\lim_n r'_n=1_{r_0(\Gamma)}$.

PROOF. Let $\{\gamma_1, \cdots, \gamma_{2m}\}$ be a set of generators of Γ such that $\gamma_i^{-1}=\gamma_{i+m}$, $1\leqq i\leqq m$. Denote $r_i(\gamma_j)$ by $\gamma_j^{(i)}$, $1\leqq j\leqq 2m$, $i=1, 2, \cdots$. By the same argument as used in Theorem 4 [7], we know that there are finitely many relations $R_1, \cdots, R_N$ satisfied by $(\gamma_1^{(0)}, \cdots, \gamma_{2m}^{(0)})$ such that for any $(x_1, \cdots, x_{2m})\in G^{2m}$ satisfying $R_1, \cdots, R_n$ and close enough to $(\gamma_1^{(0)}, \cdots, \gamma_{2m}^{(0)})$, the map $\gamma_j^{(0)}\to x_j$ $(1\leqq j\leqq 2m)$ extends to a homomorphism of $r_0(\Gamma)$ into G. Since the $r_n(\Gamma)$ are uniformly discrete and $\lim_i \gamma_j^{(i)}=\gamma_j^{(0)}$, $1\leqq j\leqq 2m$, one sees easily that $\gamma_j^{(0)}\to\gamma_j^{(i)}$, $1\leqq j\leqq 2m$ extend to homomorphisms r'_i: $r_0(\Gamma)\to G$ for large i. Now the lemma follows immediately.

BIBLIOGRAPHY

1. A. Borel, *Density properties of certain subgroups of semisimple groups without compact factor*, Ann. of Math. (2) **72** (1960), 179–188.

2. J. M. G. Fell, *Weak containment and induced representation of groups*, Canad. J. Math. **14** (1962), 237–268.

3. D. A. Kazhdan, *Connection of the dual space of a group with the structure of its closed subgroups*, Funkcional. Anal. i Priložen **1** (1967), 71–74.

4. G. W. Mackey, *Induced representation of locally compact groups*, I. Ann. of Math. (2) **55** (1962), 101–139.

5. H. C. Wang, *On a maximality property of discrete subgroup, with fundamental domain of finite measure*, Amer. J. Math. **89** (1967), 124–132.

6. S. P. Wang, *The dual space of semi-simple Lie groups*, Amer. J. Math. (to appear).

7. ———, *Limit of lattices in a Lie Group*, Trans. Amer. Math. Soc. **133** (1968), 519–526.

INSTITUTE FOR ADVANCED STUDY