APPROXIMATE EVALUATION OF A CLASS OF WIENER INTEGRALS¹

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1. It is well known that for every $\alpha \in (0, \frac{1}{2})$ almost every element x of the Wiener space C_w consisting of all the real valued functions $x(t), t \in [0, 1]$, with $x(0) = 0$ satisfies the Hölder condition $|x(t') - x(t'')| \leq h |t' - t''|^\alpha$ for some $h > 0$ which depends on x and α . Let $\phi_\alpha[x]$ be the infimum of all such $h > 0$ for fixed x and α . In [7] we showed that for every $\alpha \in (0, \frac{1}{6})$ the functional $\phi_\alpha^p[x]$ is Wiener integrable for every $p \geq 0$ and in fact

$$\begin{aligned} \int_{C_w} \phi_\alpha^p[x] d_w x &\leq 2^p (1 - 2^{-\alpha})^{-p} \\ &\quad + \left(\frac{2}{\pi}\right)^{1/2} 2^{3\alpha-1/2} (1 - 2^{3\alpha-1/2})^{-1} 2^p (1 - 2^{-\alpha})^{-p} \\ &\quad \cdot \sum_{m=1}^{\infty} (m+1)^p m^{-(m+1)} < \infty. \end{aligned}$$

We then applied this result to estimate errors in approximating Wiener integrals of a class of functionals by Lebesgue integrals in Euclidean spaces. In the present paper we apply this method to yet another class of Wiener integrals. Our result is the following:

THEOREM. *Let $f(t)$ be real valued for $t \in [0, 1]$ and satisfy*

$$(1) \quad |f(t') - f(t'')| \leq C |t' - t''|^\gamma$$

where $C, \gamma > 0$ and let $A = \max_{[0,1]} |f(t)|$. Let $g(u)$ be real valued for all real u and satisfy

$$(2) \quad |g(u') - g(u'')| \leq B |u' - u''|$$

with $B > 0$. Then the functionals $F[x]$ and $F_n[x]$ defined on C_w by

$$(3) \quad F[x] = g \left\{ \int_0^1 f(t) x^2(t) dt \right\}$$

$$(4) \quad F_n[x] = g \left\{ \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) x^2\left(\frac{k}{n}\right) \right\}, \quad n = 1, 2, \dots$$

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are Wiener measurable, and for $\alpha \in (0, \frac{1}{6})$

$$(5) \quad \left| \int_{C_w} F[x] d_w x - \int_{C_w} F_n[x] d_w x \right| \leq \frac{2BC}{(\gamma + 1)n^\gamma} + \frac{2^{3/2}AB}{(\alpha + 1)n^\alpha} \left\{ \int_{C_w} \phi_\alpha^2[x] d_w x \right\}^{1/2}.$$

The Wiener integral $\int_{C_w} F_n[x] d_w x$ which approximates $\int_{C_w} F[x] d_w x$ can be evaluated as a Lebesgue integral in the n -dimensional Euclidean space according to the well-known fact that if $G(\xi_1, \dots, \xi_n)$ is a Lebesgue measurable function defined on the n -dimensional Euclidean space and $0 = t_0 < t_1 < \dots < t_n \leq 1$ then $G[x(t_1), \dots, x(t_n)]$ is Wiener measurable and furthermore

$$\int_{C_w} G[x(t_1), \dots, x(t_n)] d_w x = \left\{ (2\pi)^n \sum_{k=1}^n (t_k - t_{k-1}) \right\}^{-1/2} \cdot \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} G(\xi_1, \dots, \xi_n) \exp \left\{ - \sum_{k=1}^n \frac{(\xi_k - \xi_{k-1})^2}{2(t_k - t_{k-1})} \right\} d\xi_1 \dots d\xi_n$$

in the sense that the existence of one side implies that of the other and the equality of the two. Thus for our $F_n[x]$ we have

$$\int_{C_w} F_n[x] d_w x = (2\pi)^{-n/2} n^{n/2} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g \left\{ \frac{1}{n} \sum_{k=1}^n f \left(\frac{k}{n} \right) \xi_k^2 \right\} \exp \left\{ - \frac{n}{2} \sum_{k=1}^n (\xi_k - \xi_{k-1})^2 \right\} d\xi_1 \dots d\xi_n.$$

2. In proving our theorem we need to estimate the Wiener integral

$$\int_{C_w} |||x|||^2 d_w x \quad \text{where } |||x||| = \max_{[0,1]} |x(t)|.$$

In [2] P. Erdős and M. Kac showed that if $X_1, X_2, X_3, \dots$ are independent identically distributed random variables each having mean value 0 and standard deviation 1 and if $s_k = X_1 + X_2 + \dots + X_k$ then

$$\lim_{n \rightarrow \infty} \text{prob.} \{ \max(s_1, s_2, \dots, s_n) < \alpha n^{1/2} \} = \sigma(\alpha)$$

where $\sigma(\alpha) = 0$ for $\alpha \leq 0$ and

$$\sigma(\alpha) = \left(\frac{2}{\pi} \right)^{1/2} \int_0^\alpha \exp \left\{ - \frac{u^2}{2} \right\} du \quad \text{for } \alpha \geq 0.$$

From this follows immediately

THEOREM A. *Let $f(u)$ be measurable on $[0, \infty)$. Then if either of the following integrals exists, both exist and they are equal:*

$$\int_{C_w} f[\max x(t)] d_w x = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty f(u) \exp\left\{-\frac{u^2}{2}\right\} du.$$

COROLLARY B. *Let $f(u)$ be measurable and nonnegative on $[0, \infty)$. Then*

$$\begin{aligned} \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty f(u) \exp\left\{-\frac{u^2}{2}\right\} du &\leq \int_{C_w} f(|||x|||) d_w x \\ &\leq 2 \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty f(u) \exp\left\{-\frac{u^2}{2}\right\} du \end{aligned}$$

where the left-hand inequality holds provided that $f(u)$ is monotonically increasing.

Corollary B follows from Theorem A and the observation that $|u| = \max\{u, -u\}$ so that

$$|||x||| = \max\left\{\max_{[0,1]} x(t), \max_{[0,1]} [-x(t)]\right\}$$

and hence if $f(u) \geq 0$ (and for the left-hand inequality, f is also monotonically increasing),

$$f[\max x(t)] \leq f(|||x|||) \leq f[\max x(t)] + f[\max[-x(t)]]$$

so that finally

$$\begin{aligned} \int_{C_w} f[\max x(t)] d_w x &\leq \int_{C_w} f(|||x|||) d_w x \\ &\leq \int_{C_w} f[\max x(t)] d_w x + \int_{C_w} f[\max[-x(t)]] d_w x \\ &= 2 \int_{C_w} f[\max x(t)] d_w x. \end{aligned}$$

3. Proof of the theorem. From the continuity of $g(u)$, the function

$$g\left\{\frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \xi_k^2\right\}$$

is continuous on the n -dimensional Euclidean space with elements $(\xi_1, \dots, \xi_n)$ and hence $F_n[x]$ as defined by (4) is Wiener measurable.

Since $x(t)$ and $f(t)$ are continuous the Riemann sum in (4) converges to the Riemann integral in (3) as $n \rightarrow \infty$. Then from the continuity of $g(u)$, $\lim_{n \rightarrow \infty} F_n[x] = F[x]$ for every $x \in C_w$. This establishes the Wiener measurability of $F[x]$. Now

$$\begin{aligned}
 & \left| \int_0^1 f(t)x^2(t)dt - \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right)x^2\left(\frac{k}{n}\right) \right| \\
 & \leq \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left| f(t)x^2(t) - f\left(\frac{k}{n}\right)x^2\left(\frac{k}{n}\right) \right| dt \\
 & \leq \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left| f(t)x^2(t) - f\left(\frac{k}{n}\right)x^2(t) \right| dt \\
 & \quad + \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left| f\left(\frac{k}{n}\right)x^2(t) - f\left(\frac{k}{n}\right)x^2\left(\frac{k}{n}\right) \right| dt \\
 & \leq |||x|||^2 \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left| f(t) - f\left(\frac{k}{n}\right) \right| dt \\
 & \quad + 2A |||x||| \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left| x(t) - x\left(\frac{k}{n}\right) \right| dt \\
 & \leq C |||x|||^2 \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \left(\frac{k}{n} - t \right)^\gamma dt \\
 & \quad + 2A |||x||| \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \phi_\alpha[x] \left(\frac{k}{n} - t \right)^\alpha dt \\
 & \leq \frac{C}{(\gamma+1)n^\gamma} |||x|||^2 + \frac{2A}{(\alpha+1)n^\alpha} |||x||| \phi_\alpha[x].
 \end{aligned}$$

Thus by (2)

$$\begin{aligned}
 |F[x] - F_n[x]| &= \left| g \left\{ \int_0^1 f(t)x^2(t)dt \right\} - g \left\{ \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right)x^2\left(\frac{k}{n}\right) \right\} \right| \\
 &\leq B \left| \int_0^1 f(t)x^2(t)dt - \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right)x^2\left(\frac{k}{n}\right) \right| \\
 &\leq \frac{BC}{(\gamma+1)n^\gamma} |||x|||^2 + \frac{2AB}{(\alpha+1)n^\alpha} |||x||| \phi_\alpha[x]
 \end{aligned}$$

and hence

$$\left| \int_{C_w} F[x] d_w x - \int_{C_w} F_n[x] d_w x \right| \leq \frac{BC}{(\gamma + 1)n^\gamma} \int_{C_w} |||x|||^2 d_w x \\ + \frac{2AB}{(\alpha + 1)n^\alpha} \left\{ \int_{C_w} |||x|||^2 d_w x \right\}^{1/2} \left\{ \int_{C_w} \phi_\alpha^2[x] d_w x \right\}^{1/2}.$$

Finally according to Corollary B

$$\int_{C_w} |||x|||^2 d_w x \leq 2 \left(\frac{2}{\pi} \right)^{1/2} \int_0^\infty u^2 \exp \left\{ -\frac{u^2}{2} \right\} du = 2.$$

This completes the proof of (5).

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