SEPARABILITY, THE COUNTABLE CHAIN CONDITION AND THE LINDELÖF PROPERTY IN LINEARLY ORDERABLE SPACES

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1. Introduction. It is known that, in metric spaces, separability, the Lindelöf property and the countable chain condition¹ (abbreviated CCC) are each equivalent to second countability. Furthermore, any space which is separable or hereditarily Lindelöf satisfies the CCC (though not conversely). In this paper, we examine separability, the Lindelöf property and the CCC in linearly orderable topological spaces (abbreviated LOTS), i.e., spaces whose topology is the open interval topology of some linear ordering of the underlying set. We shall prove that if X is a LOTS, then

- (A) X satisfies the CCC if and only if X is hereditarily Lindelöf;
- (B) X is separable if and only if X is hereditarily separable.

Result (A) was obtained for connected LOTS by T. Inagaki ([5] and [6]) and somewhat later by G. Kurepa [9]. The methods of Inagaki and Kurepa rely heavily on connectedness; our proofs avoid this assumption. (Also, see Remark 4.3, below.) Result (B) was announced without proof by G. Kurepa in [9]; L. Skula provided a proof in [14], but our proof is shorter.

The following notation will be convenient. If $(X, \leq)$ is a linearly ordered set and if $a \in X$, then $] - \infty, a[= \{x \in X \mid x < a\}$ and $] - \infty, a] = \{x \in X \mid x \leq a\}$. The sets $]a, \infty[$ and $[a, \infty[$ are defined analogously.

2. The countable chain condition in LOTS.² The following lemma is easily proved by transfinite induction.

(2.1) **LEMMA.** *Suppose X is a LOTS satisfying the CCC and suppose $Y \subseteq X$. Let $z \in Y$. Then there are countable subsets P and Q of $] - \infty, z] \cap Y$ and $[z, \infty[\cap Y$ respectively such that if $y \in Y$, then there are points $p \in P$ and $q \in Q$ such that $p \leq y \leq q$.*

(2.2) **THEOREM.** *A LOTS satisfying the CCC is hereditarily Lindelöf.*

PROOF. Suppose that X is a LOTS satisfying the CCC. To show that X is hereditarily Lindelöf, it will suffice to show that if $\mathcal{U}$ is any collection of open intervals in X , then there is a countable subcollection

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¹ A space X satisfies the *countable chain condition* if any disjoint collection of open sets is countable.

² The authors are indebted to Mary Ellen Rudin for suggesting the approach used in this section.

tion $\mathfrak{u} \subseteq \mathfrak{v}$ which covers the set $\bigcup \mathfrak{v}$ (cf. [4, p. 141, exercise B]).

Let $Y = \bigcup \mathfrak{v}$. For each $x \in Y$, let $I(x) = \{y \in Y \mid \text{the set of all points of } Y \text{ lying between } x \text{ and } y \text{ can be covered by a countable subcollection of } \mathfrak{v}\}$. It is clear that each set $I(x)$ is open in X and that for any two points $x, y \in Y$, either $I(x) = I(y)$ or else $I(x) \cap I(y) = \emptyset$. Because X satisfies the CCC, the collection $\mathcal{I} = \{I(x) \mid x \in Y\}$ is countable, say $\mathcal{I} = \{I(x) \mid x \in C\}$ where C is some countable subset of Y .

Fix $x \in C$. Using (2.1), find countable subsets $P(x)$ and $Q(x)$ of $I(x) \cap]-\infty, x]$ and $I(x) \cap [x, \infty[$ respectively such that if $y \in I(x)$, then there are points $p \in P(x)$ and $q \in Q(x)$ such that $p \leq y \leq q$. By definition of $I(x)$, for each $p \in P(x)$ and $q \in Q(x)$ there are countable subcollections $\mathfrak{v}(x, p)$ and $\mathfrak{v}(x, q)$ of $\mathfrak{v}$ which cover the sets $Y \cap [p, x]$ and $Y \cap [x, q]$ respectively. Let $\mathfrak{u} = \bigcup \{\mathfrak{v}(x, r) \mid r \in P(x) \cup Q(x) \text{ and } x \in C\}$. Then $\mathfrak{u}$ is a countable subcollection of $\mathfrak{v}$ which covers Y .

3. Separability in LOTS. Lemma (3.1) follows from (2.2) above; alternatively, it may be deduced from Lemma (3.2) below (which can be proved by modifying the proof that a LOTS is normal given in [3, p. 39]).

(3.1) **LEMMA.** *Suppose that X is a LOTS satisfying the CCC and that Y is a discrete subspace of X (discrete in the relative topology). Then Y is countable.*

(3.2) **LEMMA.** *Any LOTS is hereditarily collectionwise normal.³*

(3.3) **THEOREM.** *A separable LOTS is hereditarily separable.*

PROOF. Suppose that X is a separable LOTS and that $A \subseteq X$. Let $I(A) = \{a \in A \mid \{a\} \text{ is relatively open in } A\}$. By (3.1), $I(A)$ is countable. Let D be a countable dense subset of X . Let

$$\mathfrak{D} = \{]r, s[\mid r, s \in D, r < s, \text{ and } A \cap]r, s[\neq \emptyset\}.$$

For each interval $J \in \mathfrak{D}$, choose a point $a(J) \in A \cap J$. Let $D(A) = I(A) \cup \{a(J) \mid J \in \mathfrak{D}\}$. Clearly, $D(A)$ is a countable subset of A . To prove that $D(A)$ is dense in A , it will suffice to show that if U is open in X and if $A \cap U \neq \emptyset$, then $D(A) \cap U \neq \emptyset$. Choose $a \in A \cap U$. If $a \in I(A)$, then $a \in D(A) \cap U$. If $a \notin I(A)$, then either

- (i) $] -\infty, a[\neq \emptyset$ and $]x, a[\cap A \neq \emptyset$ whenever $x < a$, or else
- (ii) $]a, \infty[\neq \emptyset$ and $]a, y[\cap A \neq \emptyset$ whenever $y > a$.

We consider only the first case. Since U is a neighborhood of a in X , there is a point $z < a$ such that $]z, a[\subseteq U$. Applying (i), we choose

³ *Added in proof.* A detailed proof of this result by Professor L. A. Steen will appear in these Proceedings.

points $b \in]z, a[\cap A$; $c \in]b, a[\cap A$; and $d \in]c, a[\cap A$. Since D is dense in X and since $]z, c[\neq \emptyset \neq]c, a[$, there are points $r \in]z, c[\cap D$ and $s \in]c, a[\cap D$. Letting $J =]r, s[$, we obtain an element of $\mathfrak{D}$. Then $a(J) \in]z, a[\cap D(A) \subseteq U \cap D(A)$.

4. Examples. In this section, we will attempt to show how our results can be applied in general topology.

(4.1) **EXAMPLE.** Souslin spaces. Recall that a *Souslin space* is a nonseparable LOTS which satisfies the CCC. It was recently proved that the existence of a Souslin space is independent of the axioms of set theory ([10] and [15]). It follows easily from a result of Mary Ellen Rudin in [13] that if there is a Souslin space, then there is a compact, connected Souslin space. Together with our Theorem (2.2), this shows that Problem 4.7 of [1] cannot be answered affirmatively. Similarly, Problem 4.6 of [1] cannot be answered affirmatively, since Ponomarev [12] and Bennett [2] have proved that if there is a Souslin space, then there is a Souslin space with a point-countable base.

(4.2) **EXAMPLE.** Let X be the space $R \times \{0, 1\}$ with the lexicographic order topology: see [8] for details. (X is called the *double line*.) It is clear that X is a separable LOTS. Hence, X is hereditarily separable and hereditarily Lindelöf.

Let $\mathfrak{L}$ be the topology on R obtained by taking the collection $\{[a, b[\mid a, b \in R \text{ and } a < b\}$ as a base. The space $(R, \mathfrak{L})$, called the *Sorgenfrey line*, is homeomorphic to a subspace (the “top line of points”) of the double line. It follows from (2.2) and (3.3) that $(R, \mathfrak{L})$ is hereditarily separable, hereditarily Lindelöf and perfectly normal (cf. [8, p. 134, Example K]). It should be pointed out that $(R, \mathfrak{L})$ is not itself a LOTS [11].

(4.3) **REMARK.** We observe that the Souslin space with a point-countable base mentioned in (4.1) cannot be (locally) connected by [2, Theorem 15]. Also, the double line of (4.2) cannot be embedded in a (locally) connected LOTS satisfying the CCC. These two facts show that the results of Inagaki and Kurepa are inadequate for certain applications.

(4.4) **REMARK ON A PAPER OF F. B. JONES.** Professor F. Burton Jones has asked the authors to point out an error which appeared in one of his papers [7]. The five conditions listed on p. 627 are not equivalent, even in a connected LOTS, as our examples and the example given in [8, p. 164, Example K] show. In any LOTS, Jones' property (2) is equivalent to the CCC and to the hereditary Lindelöf property. In a connected LOTS, Jones' properties (1), (4) and (5) are equivalent.

5. Generalizations. In this section, m denotes an infinite cardinal number.

(5.1) DEFINITION. A space X is m -separable if X contains a dense subset having cardinality $\leq m$; X satisfies the m -chain condition if any disjoint collection of open subsets of X has cardinality $\leq m$; X is m -Lindelöf if every open cover of X has a subcover with cardinality $\leq m$.

Using the methods of §2 and §3, we can prove

(5.2) THEOREM. A LOTS satisfies the m -chain condition if and only if it is hereditarily m -Lindelöf.

(5.3) THEOREM. An m -separable LOTS is hereditarily m -separable.

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