SOME PROBLEMS CONCERNING BASES IN BANACH SPACES

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1. Introduction. A basis (x_i, f_i) for a Banach space X having the property that $0 < \inf_i \|x_i\| \leq \sup_i \|x_i\| < +\infty$ is said to be

- (1) of type P [4] provided $\sup_n \left\| \sum_{i=1}^n x_i \right\| < +\infty$,
- (2) of type P^* [4] provided $\sup_n \left\| \sum_{i=1}^n f_i \right\| < +\infty$,
- (3) of type swc_0 [2] if the sequence $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ weakly convergent to zero.

- (4) of type $(\text{swc}_0)^*$ [2] if the sequence $\{f_n\}$ in X^* is of type swc_0 .

Foiaş and Singer [2] have asked the following questions:

(Q1) If (x_i) is a basis for a Banach space X such that 0 is not a weak limit point of the set $\{x_i\}$, must there exist an f in X^* for which $|f(x_i)| \geq 1$ for all i ?

(Q2) Does $L^1[0, 1]$ have a basis of type P ?

(Q3) Does $L^1[0, 1]$ have a basis of type swc_0 ?

(Q4) Does $C[0, 1]$ have a basis of type $(\text{swc}_0)^*$?

The purpose of this paper is to give a negative answer to (Q1) and affirmative answers to (Q2)–(Q4).

2. The question (Q1). We will need the following notion of “block perturbation basis” introduced by Pełczyński and Singer [3].

(2.1) DEFINITION [3]. Let (w_i) be a basis for a Banach space E such that $\inf_i \|w_i\| > 0$. A *block perturbation* of (w_i) is any sequence (V_k) in E of the form

$$\begin{aligned} V_k &= w_k \quad \text{for } k \neq p_n \\ &= w_{p_n} + u_n \quad \text{for } k = p_n \quad (n = 1, 2, \dots), \end{aligned}$$

where $u_n = \sum_{i=m_{n-1}+1}^{p_n-1} a_i w_i + \sum_{i=p_n+1}^{m_n} a_i w_i$ with $\|u_n\| \leq M$ for $n = 1, 2, \dots$, $\{m_n\}$ and $\{p_n\}$ are increasing sequences of positive integers such that $m_0 = 0$, $m_{n-1} + 1 \leq p_n \leq m_n$, and (a_i) is a sequence of scalars.

The following result is due to Pełczyński and Singer [3].

(2.2) THEOREM. If (w_i) is a basis for E such that $\inf_i \|w_i\| > 0$, then every block perturbation (V_k) of (w_i) is a basis for E .

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Let c denote the space of all convergent sequences with the sup norm. It is well known that c is isomorphic to the space $C(\hat{I})$ (where $\hat{I}$ denotes the one point compactification of the positive integers) and that the sequence $x_1 = (1, 1, 1, 1, \dots)$, $x_2 = (0, 1, 1, 1, \dots)$, $x_3 = (0, 0, 1, 1, 1, \dots)$, $\dots$, is a basis for c .

It follows that the dual space of c may be identified with those sequences $\mu = (\mu_1, \mu_2, \dots, \mu_\infty)$ for which $\sum_i |\mu_i| + |\mu_\infty| < \infty$ (we denote by μ_∞ the μ -measure of the ideal element of $\hat{I}$). If $\mu \in c^*$ and $a = (a_i) \in c$, then $\mu(a) = \sum_i \mu_i a_i + \mu_\infty \cdot \lim_i a_i$. In particular, $\mu(x_n) = \sum_{i \leq n} \mu_i + \mu_\infty$.

It is easy to see that the sequence (y_i) defined by

$$y_1 = (x_1, 0), y_2 = (0, x_1), y_3 = (0, x_2), y_4 = (x_2, 0), y_5 = (x_3, 0), \\ y_6 = (0, x_3), y_7 = (0, x_4), \dots, \text{ is a basis for } c \oplus c.$$

Let (a_n) be any dense sequence in $[-1, 1]$. Applying Theorem (2.2), we see that the sequence (z_i) defined by

$$z_1 = (x_1, 0), z_2 = (a_1 x_1, x_1), z_3 = (0, x_2), z_4 = (x_2, a_1 x_1), \\ z_5 = (x_3, 0), z_6 = (a_2 x_3, x_3), z_7 = (0, x_4), z_8 = (x_4, a_2 x_4), \\ z_9 = (x_5, 0), z_{10} = (a_3 x_5, x_5), \dots \text{ is a basis for } c \oplus c.$$

Moreover, if $\mu_0 = (0, 0, \dots, 1) \in c^*$, then $f = (\mu_0, 0)$ and $g = (0, \mu_0)$ are elements of $(c \oplus c)^*$. Clearly, no basis element z_i is contained in the weak 0-neighborhood $U[0; f, g; \frac{1}{2}]$.

However, if $(\mu, \lambda) \in (c \oplus c)^*$, then

$$\liminf_{n \rightarrow \infty} |(\mu, \lambda)[z_{4n+2}]| = \liminf |a_n \mu_\infty + \lambda_\infty|$$

and

$$\liminf_{n \rightarrow \infty} |(\mu, \lambda)[z_{4n}]| = \liminf |\mu_\infty + a_n \lambda_\infty|.$$

Since the sequence (a_n) is dense in $[-1, 1]$, one of these two numbers is zero (for each fixed (μ, λ)).

Thus 0 is not a weak limit point of the basis (z_i) and there does not exist an $f \in (c \oplus c)^*$ for which $|f(z_i)| \geq 1$ for all i . Therefore (Q1) is answered negatively.

Note. Since $c \oplus c = c_0 \oplus c_0 = c_0$, the basis constructed in this example may be chosen in the "nice" space c_0 . The "=" here meaning "is linear homeomorphic to."

3. The question (Q2). Using the "perturbation theorem" (2.2), we will construct a basis (x_i, x_i^*) having the property that (x_i) is of type

P and (x_i^*) is similar to a basis for $L^1[0, 1]$. It then follows that $L^1[0, 1]$ has a basis of type P^* [4], and hence a basis of type P [4].

Let $(z_i)_{i=1}^\infty$ denote the normalized Haar basis for $L^1[0, 1]$. By definition $z_1(t) \equiv 1$ and,

$$\begin{aligned} z_{2^k+l}(t) &= 2^k \quad \text{for } t \in \left(\frac{2l-2}{2^{k+1}}, \frac{2l-1}{2^{k+1}} \right), \\ &= -2^k \quad \text{for } t \in \left(\frac{2l-1}{2^{k+1}}, \frac{2l}{2^{k+1}} \right), \\ &= 0 \quad \text{for other } t \\ &\quad (l = 1, 2, \dots, 2^k; k = 0, 1, 2, \dots). \end{aligned}$$

It follows easily from the orthonormality of the Haar functions that the sequence of coefficient functionals (z_i^*) in $L^\infty[0, 1]$ associated with the basis (z_i) is defined by

$$\begin{aligned} z_1^* &= z_1, \\ z_{2^k+l}^* &= \frac{1}{2^k} z_{2^k+l}. \end{aligned}$$

We will perturb the basis (z_i^*) for $[z_i^*]$ according to (2.2) to get another basis (x_i) for $[z_i^*]$ which is of type P and whose coefficient functionals (x_i^*) have the same closed linear span as those of the basis (z_i^*) , i.e. $[x_i^*] = L^1[0, 1]$.

The elements of the basis (z_i^*) to be perturbed are those of the form $z_{2^m-1}^*$, for $m = 3, 4, \dots$. Since $\|z_i^*\| = 1$ for all i , and $\|\sum_{j=2^{m-1}}^{2^m-2} z_j^*\| = 1$ for all $m \geq 3$ by definition of the functions (z_j^*) , the theorem (2.2) shows that the sequence (x_i) in $[z_i^*]$ defined by

$$\begin{aligned} x_i &= z_i^* \quad \text{for } i \neq 2^m - 1, m \geq 3, \\ &= z_{2^m-1}^* - \sum_{j=2^{m-1}}^{2^m-2} z_j^* \quad \text{for } i = 2^m - 1, m \geq 3, \end{aligned}$$

is a basis for $[z_i^*]$. Moreover, if (x_i^*) denotes the sequence of coefficient functionals associated with the basis (x_i) , then an examination of the proof of (2.2) [3] shows that $[x_i^*]$ is identical to the closed linear span of the sequence of coefficient functionals associated with (z_i^*) . Since this last is just $[z_i] = L^1[0, 1]$, we have that $[x_i^*] = L^1[0, 1]$.

We claim that (x_i) is of type P in $[z_i^*]$. In fact, $\|x_i\| = 1$ for all i , and for any $n = 2^m + l$, where $1 \leq l \leq 2^m - 2$,

$$\begin{aligned} \left\| \sum_{i=1}^n x_i \right\| &\leq \left\| \sum_{i=1}^3 x_i \right\| + \left\| \sum_{i=4}^{2^m-1} x_i \right\| + \left\| \sum_{i=2^m}^{2^m+l} x_i \right\| \\ &= \left\| \sum_{i=1}^3 x_i \right\| + \left\| \sum_{j=3}^m z_{2^j-1}^* \right\| + \left\| \sum_{i=2^m}^{2^m+l} z_i^* \right\|, \end{aligned}$$

by definition of the sequence (x_i) . If $l = 2^m - 1$ the above becomes

$$\left\| \sum_{i=1}^n x_i \right\| \leq \left\| \sum_{i=1}^3 x_i \right\| + \left\| \sum_{j=3}^{m+1} z_{2^j-1}^* \right\|,$$

while if $l = 2^m$, we have

$$\left\| \sum_{i=1}^n x_i \right\| \leq \left\| \sum_{i=1}^3 x_i \right\| + \left\| \sum_{j=3}^{m+1} z_{2^j-1}^* \right\| + \|z_{2^{m+1}}\|.$$

In any case, by definition of the Haar functions, $\left\| \sum_{j=3}^p z_{2^j-1}^* \right\| = 1$ for all $p \geq 3$, and $\left\| \sum_{i=2^m}^{2^m+l} z_i^* \right\| = 1$ for all $1 \leq l \leq 2^m - 2$. Hence we see that (x_i) is of type P . It follows that (x_i^*) is similar to a basis of type P^* for $L^1[0, 1]$, and according to our previous remarks, $L^1[0, 1]$ has a basis of type P .

REMARK 1. Since a basis of type P^* is certainly weakly closed [2], it follows from the above that $L^1[0, 1]$ has a weakly closed basis. Also, since the coefficient functionals of the basis of type P for $L^1[0, 1]$ are of type P^* , it follows that $L^1[0, 1]$ has a (weakly closed)* basis [2].

These observations answer two other questions raised by Foiaş and Singer [2].

REMARK 2. The method of perturbation which was applied above to obtain a type P basis for $[z_i^*]$ will clearly also yield a type P basis for $C[0, 1]$ when applied to the usual Schauder basis for $C[0, 1]$. Foiaş and Singer have previously shown, using a different construction, that $C[0, 1]$ has a basis of type P [2].

4. The questions (Q3) and (Q4). Zippin has proved the following lemma.

(4.1) LEMMA (ZIPPIN [5]). *Let $(x_i)_{i=1}^\infty$ be a basis for a Banach space X . Assume that $y_k = \sum_{i=p_k+1}^{p_{k+1}} a_i x_i \neq 0$, where $\{p_k\}_{k=2}^\infty$ is an increasing sequence of positive integers and $p_1 = 0$. Then there exists a basis $(z_i)_{i=1}^\infty$ for X such that for each $k \geq 1$, $z_{p_{k+1}} = y_k$.*

Moreover, as Zippin's proof shows, if (z_i^*) is the sequence of coefficient functionals associated with the basis (z_i) , then $[z_i^*] = [x_i^*]$,

where (x_i^*) is the sequence of coefficient functionals associated with (x_i) .

(4.2) PROPOSITION. *Let X be a Banach space with a basis in which weak and norm convergence are not equivalent. Then X has a basis of type swc_0 .*

PROOF. Since weak and norm convergence are not equivalent there exists a sequence (w_i) in X which is weakly convergent to zero but for which $\inf_i \|w_i\| > 0$.

Let (x_i) be a basis for X . By a theorem of Bessaga and Pełczyński [1] there is a subsequence (w_{n_k}) which is a basic sequence in X similar to a block basic sequence $(y_k)_{k=1}^\infty = (\sum_{i=p_k+1}^{p_{k+1}} a_i x_i)_{k=1}^\infty$, where $p_1 = 0$ and $(p_i)_{i=2}^\infty$ is an increasing sequence of positive integers. It follows that (y_k) is weakly convergent to zero.

By Zippin's Lemma (4.1) there exists a basis (z_i) for X such that $z_{p_{k+1}} = y_k$. Since (y_k) converges weakly to zero, the basis (z_i) is clearly of type swc_0 .

The following corollary to (4.1) answers (Q3).

(4.3) COROLLARY. *The space $L^1[0, 1]$ has a basis of type swc_0 .*

PROOF. The sequence $\{\sin 2n\pi x\}$ is weakly convergent to zero in $L^1[0, 1]$ but not norm convergent. Apply Proposition (4.2).

To answer (Q4), let (x_i, x_i^*) be any basis for $L^1[0, 1]$ for which $[x_i^*]$ is isomorphic to $C[0, 1]$ (e.g. let (x_i) be the normalized Haar basis for $L^1[0, 1]$ [2]). By the above construction there exists a basis (z_i) for $L^1[0, 1]$ which is of type swc_0 and whose coefficient functionals (z_i^*) span a space isomorphic to $[x_i^*] = C[0, 1]$. Therefore, since the sequence of coefficient functionals associated with (z_i^*) is similar to (z_i) we see that $C[0, 1]$ has a basis of type $(\text{swc}_0)^*$, and (Q4) is also answered affirmatively.

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