A FIXED POINT THEOREM FOR MAPPINGS WITH A CONTRACTIVE ITERATE

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1. Let (X, d) be a metric space and $f: X \rightarrow X$ a mapping. If f satisfies the condition

$$(1) \quad d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in X$ and for some $k < 1$, then f is called a contraction.

A well-known theorem of Banach [1] states that for X complete, each contraction mapping f has a unique fixed point u , and the successive approximations $\{f^n(x_0)\}$ converge to u for each $x_0 \in X$. In this paper, we investigate mappings which are not necessarily contractions and prove the following result.

THEOREM. *Let (X, d) be a complete metric space, and $f: X \rightarrow X$ a continuous mapping satisfying the condition: there exists a $k < 1$ such that for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$*

$$(2) \quad d(f^{n(x)}(y), f^{n(x)}(x)) \leq kd(y, x).$$

Then f has a unique fixed point u and $f^n(x_0) \rightarrow u$ for each $x_0 \in X$.

LEMMA. *If $f: X \rightarrow X$ be any mapping satisfying the condition of the above theorem then for each $x \in X$, $r(x) = \sup_n d(f^n(x), x)$ is finite.*

PROOF. Let $x \in X$ and let

$$l(x) = \max\{d(f^k(x), x): k = 1, 2, \dots, n(x)\}.$$

If n is a positive integer, there exists an integer $s \geq 0$ such that $s \cdot n(x) < n \leq (s+1) \cdot n(x)$, and

$$\begin{aligned} d(f^n(x), x) &\leq d(f^{n(x)} \cdot f^{n-n(x)}(x), f^{n(x)}(x)) + d(f^{n(x)}(x), x) \\ &\leq kd(f^{n-n(x)}(x), x) + l(x) \\ &\leq l(x) + kl(x) + k^2l(x) + \dots + k^sl(x) \\ &\leq l(x)/(1 - k) \quad \text{for all } n \geq 0. \end{aligned}$$

Hence $r(x) = \sup_n d(f^n(x), x)$ is finite.

PROOF OF THE THEOREM. Let $x_0 \in X$ be arbitrary. Let $m_0 = n(x_0)$, $x_1 = f^{m_0}(x_0)$ and inductively $m_i = n(x_i)$, $x_{i+1} = f^{m_i}(x_i)$. We show that

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the sequence $\{x_n\}$ is a convergent sequence. By routine calculation we have

$$d(x_{n+1}, x_n) = d(f^{m_{n-1}} \cdot f^{m_n}(x_{n-1}), f^{m_{n-1}}(x_{n-1})) \leq kd(f^{m_n}(x_{n-1}), x_{n-1}) \leq \cdots \leq k^n d(f^{m_n}(x_0), x_0).$$

Therefore, it follows by lemma that $d(x_{n+1}, x_n) \leq k^n r(x_0)$. Thus, for $m > n$,

$$d(x_m, x_n) \leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \leq \frac{k^n}{1-k} r(x_0) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The sequence $\{x_n\}$ is therefore Cauchy. Let $x_n \rightarrow u \in X$.

If $f(u) \neq u$, then, there exists a pair of disjoint closed neighborhoods U and V such that $u \in U, f(u) \in V$ and

$$(3) \quad \rho = \inf\{d(x, y) : x \in U, y \in V\} > 0.$$

Since f is continuous, $x_n \in U$ and $f(x_n) \in V$ for all n sufficiently large. However,

$$d(f(x_n), x_n) = d(f^{m_{n-1}} \cdot f(x_{n-1}), f^{m_{n-1}}(x_{n-1})) \leq kd(f(x_{n-1}), x_{n-1}) \leq \cdots \leq k^n d(f(x_0), x_0) \rightarrow 0$$

as $n \rightarrow \infty$, contradicting (3). Thus $f(u) = u$.

The uniqueness of the fixed point follows immediately from (2). To show that $f^n(x_0) \rightarrow u$, set

$$\rho_* = \max\{d(f^m(x_0), u) : m = 0, 1, 2, \dots, (n(u) - 1)\}.$$

If n is a sufficiently large integer, then $n = r \cdot n(u) + q$, $0 \leq q < n(u)$, $r > 0$, and

$$d(f^n(x_0), u) = d(f^{r \cdot n(u) + q}(x_0), f^{n(u)}(u)) \leq kd(f^{(r-1) \cdot n(u) + q}(x_0), u) \leq \cdots \leq k^r d(f^q(x_0), u) \leq k^r \rho_*.$$

Since $n \rightarrow \infty$ implies $r \rightarrow \infty$, we have $d(f^n(x_0), u) \rightarrow 0$ as $n \rightarrow \infty$. This establishes the theorem.

2. In this section we give an example of a continuous function f which satisfies condition (2), but is not a contraction. The author is thankful to the referee for showing that no iterate of f is a contraction. Bryant [2] considers a function f (not necessarily continuous) on a complete metric space X into itself for which there exists a k , $0 \leq k < 1$, and a positive integer n , such that $d(f^n(x), f^n(y)) \leq kd(x, y)$ for all $x, y \in X$. That Bryant's condition is stronger than condition (2) is shown below. The author is grateful to Professor David R. Anderson for his help in the following example.

EXAMPLE. Let X be the closed unit interval $[0, 1]$ with the usual metric. Write

$$X = \bigcup_{n=1}^{\infty} \left[\frac{1}{2^n}, \frac{1}{2^{n-1}} \right] \cup \{0\},$$

and let $f: X \rightarrow X$ be defined as follows.

For each $n = 1, 2, \dots$, let

$$f: \left[\frac{1}{2^n}, \frac{1}{2^{n-1}} \right] \rightarrow \left[\frac{1}{2^{n+1}}, \frac{1}{2^n} \right]$$

be defined by

$$\begin{aligned} f(x) &= \frac{n+2}{n+3} \left(x - \frac{1}{2^{n-1}} \right) + \frac{1}{2^n}, & \text{if } x \in \left[\frac{3n+5}{2^{n+1}(n+2)}, \frac{1}{2^{n-1}} \right] \\ &= \frac{1}{2^{n+1}} & \text{if } x \in \left[\frac{1}{2^n}, \frac{3n+5}{2^{n+1}(n+2)} \right] \end{aligned}$$

and let $f(0) = 0$.

It is obvious that f is a nondecreasing, continuous function on $[0, 1]$ with 0 as the only fixed point, and that f is not a contraction. If $x \in [1/2^n, 1/2^{n-1}]$ and $y \in X$, then by a routine examination of cases $y \in [1/2^m, 1/2^{m-1}]$ for $m \geq n$ and $m \leq n$, it is easy to verify that f satisfies

$$|f(x) - f(y)| \leq \frac{n+3}{n+4} |x - y| \quad \text{for all } y \in X.$$

Therefore, if we choose $k = 1/2$ in (2), then for each $x \in [1/2^n, 1/2^{n-1}]$, $n(x)$ may be taken as $n+3$, whereas $n(0)$ may be taken as any integer greater than one.

To show that the condition in [2] is stronger than (2), let $0 \leq k < 1$, and N (a natural number) be given; it will be shown that there exists x and y such that $|f^N(x) - f^N(y)| > k|x - y|$. Choose and fix $n > (Nk/(1-k)) - 2$. Since f^i is uniformly continuous on $[0, 1]$ for $i = 1, 2, \dots, N$, there is some $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f^i(x) - f^i(y)| < \frac{n + N + 3}{(n + N + 2)2^{n+N+1}}$$

for $i = 1, 2, \dots, N$. Setting $x = 1/2^{n-1}$ and y any member of $[1/2^n, 1/2^{n-1}]$ such that $0 < |x - y| < \delta$, it can be shown that $f^i(x)$ and $f^i(y)$ are both members of

$$\left[\frac{3(n+i)+5}{2^{n+i+1}(n+i+2)}, \frac{1}{2^{n+i-1}} \right]$$

for $i=1, 2, \dots, N$. Thus

$$\begin{aligned} |f(x) - f(y)| &= \frac{n+2}{n+3} |x - y|, \\ |f^2(x) - f^2(y)| &= \frac{n+2}{n+4} |x - y|, \dots, |f^N(x) - f^N(y)| \\ &= \frac{n+2}{n+2+N} |x - y| > k |x - y|. \end{aligned}$$

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