TWO CONVEX COUNTEREXAMPLES: A DISCONTINUOUS ENVELOPE FUNCTION AND A NONDIFFERENTIABLE NEAREST-POINT MAPPING

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Introduction. This paper gives counterexamples for two unrelated conjectures which pertain to convexity theory.

Suppose f is any function defined on a closed convex set C in finite-dimensional Euclidean space. Its *envelope function* $\text{env } f$ is defined as the pointwise supremum of all linear functions which f dominates everywhere. It is elementary and well known that $\text{env } f$ is convex, and hence continuous in the relative interior of C . It is similarly known that $\text{env } f$ is lower-semicontinuous on the boundary of C . Witsenhausen [14] uses and studies envelope functions. He suggested the following orally:

CONJECTURE 1. If f is continuous everywhere on C , then $\text{env } f$ is also continuous everywhere on C .

I provide a very simple counterexample in 3-space which shows that this is not true. Incidentally, though the proof is not on paper, I feel fairly certain that Witsenhausen's conjecture is correct in 2-space.

If C is a closed convex set in finite-dimensional Euclidean space, then for any point x in the space there is a unique point $P(x)$ in C which is nearest to x . The mapping P is called the *projection* onto C . (Many interesting results about this mapping are given in Phelps [9], [10].) It is easy to prove and well known that P is a contraction mapping, hence continuous. It is also easy to see that P need not have much in the way of differentiability properties, for if C is a polyhedron in 2-space, then it is easy to find points and directions at which P does not even have a two-sided directional derivative. The existence in general of a one-sided directional derivative, however, is not so easy to decide. This question does not seem to be raised in such natural places as Eggleston [4] and Valentine [11].

CONJECTURE 2. For all points and directions, P has a one-sided directional derivative.

My mildly complicated counterexample is a set C (in 3-space), which is the convex hull of a countable infinity of points. I suspect that the conjecture is correct in 2-space. In n -space it would be interesting to know whether the set of points and directions at which the conjecture holds is dense in $E_n \times S_{n-1}$. If not, the convex set would have to be extremely jagged!

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Counterexample to Conjecture 1. Envelope functions are used by Hans Witsenhausen [14] and by other authors under other names (for example, in an important paper J. J. Moreau [7] calls it the “ Γ -regularized of f ”). Suppose f is any function defined on some closed convex set C in E_n . We say that a (nonhomogeneous) linear function

$$u(x) = a + \sum_1^n b_i x_i$$

is *dominated* by f if $u(x) \leq f(x)$ for all x in C . Let $L(f)$ be the set of all linear functions dominated by f , and define the envelope function $\text{env } f$ by

$$\text{env } f(x) = \sup_{u \text{ in } L(f)} \{u(x)\}.$$

The authors mentioned point out the following simple facts: Because $\text{env } f$ is the supremum of convex (even linear) functions, it is convex, hence continuous on the (relative) interior of C . As the supremum of continuous functions, $\text{env } f$ must be lower-semicontinuous everywhere, which is pertinent on the (relative) boundary of C .

In his paper, Witsenhausen faces the situation where the original function f is known to be continuous. For his control theory application, he wished to prove continuity of $\text{env } f$ on the boundary of C . He was able to prove (Proposition 31 on p. 20 of [14]) only two special cases: if C is a convex polyhedron, that is, the convex hull of a finite set, then $\text{env } f$ is continuous on C ; or if x is an exposed boundary point of C , that is, if there is a hyperplane which meets C only in x , then $\text{env } f$ is continuous at x . However, he conjectured (oral communication) that continuity of f always guaranteed the continuity of $\text{env } f$.

The key to the counterexample is the structure of the “faces” of the convex set. If C is closed convex, and H is any supporting hyperplane (that is, a hyperplane which meets C but does not separate it), then the intersection of C and H is called a *face* of C . This immediately generalizes the faces of a polyhedron. An *edge* means a 1-dimensional face, and a *vertex* means a 0-dimensional face. Note that a vertex means exactly the same thing as an exposed point.

In the counterexample, C has vertices (exposed points) whose limiting point is in the relative interior of an edge. This is the central fact which makes the example work. Figure 1 displays C .

In 3-space, denote the coordinates by a , b , and c . The function f is extremely simple: $f(x) = -c^2$. The region C is the convex hull of a circle in the coordinate (a, b) -plane (call this the “equator”) and a

line segment of length 2 parallel to the c -axis, having its midpoint on the equator (call this midpoint the "special point"). Except at the special point, every point of the equator is an exposed point.

The proof that $\text{env } f$ is discontinuous at the special point is quite simple. Witsenhausen shows without difficulty (Proposition 31, proof of Case 2), that if f is continuous at an exposed point, then $\text{env } f = f$ at that point. This shows that $\text{env } f$ is 0 on the equator, excepting the special point. It also shows that $\text{env } f$ is -1 at the ends of the line segment. Since $\text{env } f$ is convex, it must never exceed -1 along the line segment. Thus $\text{env } f$ is discontinuous at the special point.

As a matter of fact, $\text{env } f(a, b, c) = -|c|$ except along the line segment, where it is -1 . Thus it is discontinuous all along this segment, except at the ends. The reader may easily verify this.

Counterexample to Conjecture 2. Moreau [7] defines the *projection* function onto a closed convex set C and calls it proj or proj_C . With Nashed (1968), we call it P . For any x in E_n , $P(x)$ is defined to be the unique point of C which is nearest to x . This definition directly generalizes projections onto a linear subspace. Moreau notes the simple fact that if x_1 and x_2 are any two points in the space, then $\text{distance}(P(x_1), P(x_2)) \leq \text{distance}(x_1, x_2)$, that is, P is a contraction mapping. As an elementary consequence, P is continuous.

In line with standard definitions, the directional derivative of P at x in the direction u (the vector u is assumed to have length 1) is given by

$$\lim_{\epsilon \rightarrow 0} \frac{P(x + \epsilon u) - P(x)}{\|(x + \epsilon u) - x\|} = \lim_{\epsilon \rightarrow 0} \frac{P(x + \epsilon u) - P(x)}{\epsilon}.$$

If ϵ is restricted to approach 0 from the positive side, then this becomes the one-sided directional derivative.

By considering convex polygons in the plane, it is easy to see that the two-sided directional derivative of P may often fail to exist. However, in such simple examples the one-sided directional derivative does exist. Must it always? This question arose quite naturally during the examination of the differentiability properties of least-squares monotone regression (for material on monotone regression, see Barton and Mallows [1], Bartholomew [2] and [3], Kruskal [5], Miles [6], and van Eeden [12] and [13]). The earliest known reference to monotone regression is Ayer, Brunk, Ewing, Ried, and Silverman [0].

The idea behind our example is this. We shall have points x_i which approach a limit x_∞ from some fixed direction. The projections

$p_i = P(x_i)$ will of course approach $p_\infty = P(x_\infty)$. However, we will arrange matters so that for *even* values of i the vector $p_i - p_\infty$ will point almost in one direction (converging to that direction as $i \rightarrow \infty$), while for *odd* values of i the vector $p_i - p_\infty$ will converge to a different direction. This is enough to assure nonexistence of the one-sided derivative. Figure 1 displays the example, with considerable liberties of scale to gain visual clarity.

FIGURE 1. Note that scale has been freely altered to improve visual clarity.

In 3-space denote the coordinates by a, b, c . To visualize the counterexample, first imagine the two curves described in parametric form by $a(s) = s, b(s) = s, c(s) = -s^2$ and $a(s) = s, b(s) = -s, c(s) = -s^2$. For some suitable number $\alpha < 1$ (later we will use $\alpha = 0.01$ and $\alpha = 0.25$), imagine the points for which $s = \alpha^i$ for $i = 1, 2, 3, \dots$. Along the first curve, we use the points for which i is even, along the second curve the points for which i is odd. Both of these sequences converge to the origin, which we add to the collection. Finally, we add the point $a = -R, b = 0, c = 0$ (with R large and positive) to the collection. The set C is the convex hull of all these points.

The point x where we prove the derivative not to exist is $a = 0$,

$b=0$, $c=1$. The projection of x on C is the origin. The direction in which the one-sided derivative does not exist is $a=1$, $b=0$, $c=0$. To see this, imagine a point y approaching x from this direction. The projection of y on C lies on the two curves alternately (as well as lying between them most of the time) for y arbitrarily close to x . (This is what requires calculation.) Thus the ratio used to define the derivative gets close to both the line $b=a$ and close to the line $b=-a$ infinitely often as y approaches x , and thus does not converge.

Define C to be the closed convex hull of the following countable collection of points: $q = [-R, 0, 0]$, $p_i = [\alpha^i, (-1)^i \alpha^i, -(\alpha^i)^2]$ for $i \geq 0$, and $p_\infty = [0, 0, 0]$, where R is either ∞ or some large positive number, and $\alpha > 0$ is some suitable small number (for example, $\alpha = 0.01$). Then $P([0, 0, 1]) = [0, 0, 0]$, and the one-sided derivative of P at $[0, 0, 1]$ in the direction $[1, 0, 0]$ does not exist.

In particular, for $\lambda > 1$, let $x_i = [\lambda \alpha^i, 0, 1]$ for $i \geq 0$, and $x_\infty = [0, 0, 1]$. Then x_i approaches x_∞ from the positive $[1, 0, 0]$ direction.

We shall establish by direct calculation that given suitable parameter values (for example, $\lambda = 10$ and $\alpha = 0.01$), $P(x_i) = p_i$ and $P(x_\infty) = p_\infty$. Then trivially,

$$\begin{aligned} & \frac{P(x_i) - P(x_\infty)}{\|x_i - x_\infty\|} \\ &= \frac{p_i - p_\infty}{\lambda \alpha^i} \\ &= (1/\lambda)[1, (-1)^i, -\alpha^i] \rightarrow \begin{cases} (1/\lambda)(1, 1, 0) & \text{if } i \text{ even and } i \rightarrow \infty, \\ (1/\lambda)(1, -1, 0) & \text{if } i \text{ odd and } i \rightarrow \infty. \end{cases} \end{aligned}$$

This shows that the limit which defines the one-sided derivative does not exist.

It only remains to verify that the p 's are the projections of the x 's for suitable parameter values. It is obvious that $P(x_\infty) = p_\infty$. To show that $P(x_i) = p_i$, it is enough to show that all the points p_j and q (which define C) lie behind the hyperplane through p_i which is normal to $p_i - x_i$. Thus it suffices to show that

$$(1) \quad (p_i - x_i) \cdot (p_i - p_j) \leq 0$$

$$(2) \quad (p_i - x_i) \cdot (p_i - q) \leq 0$$

for all i and j . The latter inequality is obvious. By direct calculation, (1) can be written as follows, where $w = \alpha^{i-j}$:

$$\alpha^{2i} \{ (1 - \lambda)(1 - w) + (1 - (-1)^{i+j}w) + (1 + \alpha^{2i})(1 - w^2) \} \leq 0.$$

Dividing by α^{2i} , and noting that $-(-1)^{i+j}w \leq +w$, we see that it suffices to show that

$$(3) \quad -\lambda(1-w) + (3-w^2) + \alpha^{2i}(1-w^2) \leq 0.$$

For $j=i$, (1) is trivial, so we assume $j \neq i$ in (3). For $j > i$, $w < 1$, so (3) becomes

$$\lambda \geq \frac{3-w^2}{1-w} + \alpha^{2i}(1+w) = \frac{2}{1-w} + (1+w)(1+\alpha^{2i}).$$

As $w = \alpha^{j-i} \leq \alpha$ in this case, and $\alpha^{2i} \leq \alpha^2$, this holds whenever

$$\lambda \geq \lambda_1 \equiv \frac{2}{1-\alpha} + (1+\alpha)(1+\alpha^2).$$

For $j < i$, $w > 1$, so (3) becomes

$$\lambda \leq \frac{w^2-3}{w-1} + \alpha^{2i}(w+1) = -\frac{2}{w-1} + (1+w)(1+\alpha^{2i}).$$

As $w = \alpha^{j-i} \geq 1/\alpha$ in this case, and $\alpha^{2i} \geq 0$, this holds whenever

$$\lambda \leq \lambda_2 \equiv -\frac{2\alpha}{1-\alpha} + \frac{1+\alpha}{\alpha}.$$

Brief numerical calculation now reveals that for values of $\alpha \leq \alpha_0 \approx 0.264$, $\lambda_1 \leq \lambda_2$ so that solutions exist. Among these solutions are $\alpha = 0.25$, $\lambda = 4$, and $\alpha = 0.01$, $\lambda = 10$.

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