THE ALGEBRA OF FUNCTIONS WITH FOURIER TRANSFORMS IN L^p

JOHN C. MARTIN AND LEONARD Y. H. YAP[1]

The purpose of this note is to prove the following

THEOREM. *Let G be a nondiscrete locally compact Abelian group with dual group $\hat{G}$. Let*

$$A_p(G) = \{f \in L^1(G) : \hat{f} \in L^p(\hat{G})\},$$

which for $p \geq 1$ is a Banach algebra under convolution, with norm

$$\|f\|^p = \|f\|_1 + \|\hat{f}\|_p.$$

Let $A_p(G)^2$ denote the ideal generated by the products $f * g$, where $f, g \in A_p(G)$. Then

(1) *$A_p(G)^2$ is a proper dense subset of $A_p(G)$.*

(2) *$A_p(G)$ contains a maximal ideal which is neither closed, prime, nor regular.*

(3) *$A_p(G)$ has no bounded approximate identity.*

(4) *There is a discontinuous positive linear functional on $A_p(G)$.*

REMARKS. These algebras were introduced in [3], and several properties similar to those of $L^1(G)$ were established. The properties studied here are in contrast to those of $L^1(G)$ (see [1], [7], [8]). By Cohen's factorization theorem [1], $L^1(G)^2 = L^1(G)$; the fact that every maximal ideal of $L^1(G)$ is closed, prime, and regular follows immediately from Cohen's theorem and the following facts.

LEMMA [5, pp. 87, 88]. *Let R be a commutative Banach algebra such that $R^2 \neq \{0\}$. Then R contains a nonprime maximal ideal if and only if $R^2 \neq R$, and in this case each nonprime maximal ideal is a maximal subspace of R containing R^2. If $1 \notin R$, then a maximal ideal is regular if and only if it is prime.*

PROOF OF THEOREM. It is clear from the lemma that (2) follows from (1). To prove (1) we show first that for $p \geq 1$, $A_p(G)^2 \subset A_q(G)$, where $q = \max(1, p-1)$. If $1 \leq p \leq 2$, and $f, g \in A_p(G)$, then $\hat{f}$ and $\hat{g} \in L^2(\hat{G})$, since they are bounded, and thus $\hat{f} \cdot \hat{g} \in L^1(G)$ by Hölder's inequality. If $p > 2$, then $2(p-1) > p$ and hence $|\hat{f}|^{p-1}$ and $|\hat{g}|^{p-1}$

Received by the editors May 19, 1969 and, in revised form, July 22, 1969.

[1] Supported by a NASA fellowship and the Air Force Office of Scientific Research under AFOSR grant number 68-1577, respectively.

belong to $L^2(\hat{G})$. Next note that $A_1(G)^2 \subset A_{1/2}(G)$.

Now we fix $p \geqq 1$ and suppose $A_p(G)^2 = A_p(G)$; i.e., if f is any element of $A_p(G)$, f may be written as

$$f = \sum_{i=1}^{n} \alpha_i(f_i * f_i') \qquad (\alpha_i \in \mathbf{C}, f_i, f_i' \in A_p(G)).$$

Representing each f_i in the same manner, and continuing this process, it is clear that for any positive integer m, we may write f as

$$f = \sum_{i=1}^{N} \alpha_i(f_{i,1} * f_{i,2} * \cdots * f_{i,m}) \qquad (f_{i,j} \in A_p(G)).$$

Letting m be an integer greater than 2^p, we see that $A_p(G) \subset A_1(G)$ and $A_1(G) \subset A_{1/2}(G)$. To complete the proof of (1) it therefore suffices to construct, for $p > 1$ a function f which is in $A_p(G)$ and not in $A_1(G)$, and for $p = 1$ a function f in $A_1(G)$ but not in $A_{1/2}(G)$.

Since G is nondiscrete, $\hat{G}$ is noncompact, and we may choose a symmetric neighborhood U of the identity in $\hat{G}$ whose closure is compact, and a sequence $\gamma_1, \gamma_2, \gamma_3, \cdots \in \hat{G}$ such that $\gamma_i U^2 \cap \gamma_j U^2 = \varnothing$ if $i \neq j$. Let $a_k = k^{-1}$ if $p > 1$, and $a_k = k^{-1/2}$ if $p = 1$, and define $g = \chi_U$, (the characteristic function of U); $h = \sum_{k=1}^{\infty} a_k \chi_{\gamma_k U^2}$. Then g and h are in $L^2(\hat{G})$ and hence $g * h = \hat{f}$, for some $f \in L^1(G)$, by [6, Theorem 1.6.3]. Now observe that $g \in L^1(\hat{G})$ and $h \in L^p(\hat{G})$, so $g * h \in L^p(\hat{G})$, and $g * h(\tau) = a_k \rho(U)$ for $\tau \in \gamma_k U$ (ρ is the Haar measure of $\hat{G}$). Finally by considering separately the cases $p > 1$ and $p = 1$, we see that $A_p(G)^2$ is a proper subset of $A_p(G)$.

To see that $A_p(G)^2$ is dense in $A_p(G)$, let $f \in A_p(G)$ and $\epsilon > 0$; choose an approximate identity $\{u_\alpha\}$ in $L^1(G)$ such that $\|u_\alpha\|_1 \leqq 1$ and $\hat{u}_\alpha$ has compact support for each α. Pick a compact set $K \subset \hat{G}$ so that $\int_{\hat{G} \setminus K} |\hat{f}|^p d\rho < \epsilon^p 2^{-(2p+1)}$, and choose α_1 so that $\alpha > \alpha_1$ implies $\|f - u_\alpha * f\|_1 < \min(\epsilon/2, (\epsilon/4)\rho(K)^{-1/p})$. Then a direct computation shows that for $\alpha > \alpha_1$, $\|f - u_\alpha * f\|^p \equiv \|f - u_\alpha * f\|_1 + \|\hat{f} - \hat{u}_\alpha \hat{f}\|_p < \epsilon$. Since $f * u_\alpha \in A_p(G)^2$, the proof of (1) is complete.

To prove (3), note that for f, $g \in A_p(G)$, we have $\|f * g\|^p \leqq \|f\|_1 \cdot \|g\|^p$. The existence of a bounded approximate identity $\{g_\alpha\}$ would imply the existence of a constant C such that $\|f\|^p \leqq C\|f\|_1$ for each $f \in A_p(G)$. $A_p(G)$ would then be a closed subspace of $L^1(G)$, and hence equal to $L^1(G)$ since it is dense. But this is clearly not the case.

Finally, part (4) follows from the existence of a nonclosed maximal ideal in $A_p(G)$; this completes the proof of the theorem.

REMARK. The factorization theorem of Hewitt [2] provides an immediate proof of (3) and the density of $A_p(G)^2$ in $A_p(G)$. The simple proof of (3) given here was suggested to us by a comment of James Burnham.

The authors wish to thank the referee for helpful suggestions, especially about the organization of the material presented here.

REFERENCES

1. P. J. Cohen, *Factorization in group algebras*, Duke Math. J. **26** (1959), 199–205. MR **21** #3729.

2. E. Hewitt, *The ranges of certain convolution operators*, Math. Scand. **15** (1964), 147–155. MR **32** #4471.

3. R. Larsen, T. S. Liu, and J. K. Wang, *On functions with Fourier transforms in* L_p, Michigan Math. J. **11** (1964), 369–378. MR **30** #412.

4. P. Porcelli and H. S. Collins, *Ideals in group algebras*, Studia Math. (to appear). Research announcement in Bull. Amer. Math. Soc. **75** (1969), 83–84.

5. P. Porcelli, *Linear spaces of analytic functions*, McGraw-Hill, New York, 1966.

6. W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Appl. Math., no. 12, Interscience, New York, 1962. MR **27** #2808.

7. N. Th. Varopoulos, *Sur les formes positives d'une algèbre de Banach*, C. R. Acad. Sci. Paris **258** (1964), 2465–2467. MR **33** #3121.

8. ———, *Continuité des formes linéaires positives sur une algèbre de Banach avec involution*, C. R. Acad. Sci. Paris **258** (1964), 1121–1124. MR **28** #4387.

RICE UNIVERSITY