AN ERGODIC THEOREM FOR SEMIGROUPS¹

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Introduction. The purpose of this paper is to prove a maximal ergodic theorem (Theorem 1) for continuous semigroups of positive contractions. This result is a continuous extension of the maximal ergodic theorem first introduced by Brunel [4] and then generalized in [1] and [2]. The continuous version of Chacon-Ornstein's ratio ergodic theorem which can be proved directly [7], as noted by the referee, is also a corollary of our result. This continuous version of Chacon-Ornstein's theorem was also proved by Berk [3] by different methods.

Definitions and preliminaries. Let $(X, \mathfrak{F}, \mu)$ be a σ -finite measure space, $(R^+, \mathcal{L}, m)$ the Lebesgue measure space on $R^+ = [0, \infty)$ and $L_1 = L_1(X, \mathfrak{F}, \mu)$ the Banach space of equivalence classes of integrable real valued functions on X . The elements of L_1 will be identified in the usual manner and L_1^+ will denote the class of nonnegative L_1 functions.

Let $T = \{T_t : t \geq 0\}$ denote a semigroup of positive contractions from L_1 to L_1 , i.e. $T_0 = I$ the identity operator, $T_{t+s} = T_t T_s$, $T_t L_1^+ \subset L_1^+$, T_t is a linear operator and

$$\int_X |T_t f| d\mu \leq \int_X |f| d\mu$$

for all $s, t \geq 0$ and all $f \in L_1$.

We shall assume that T is a strongly continuous semigroup. This means that for every $f \in L_1$, $T_{(\cdot)} f$ is continuous in L_1 norm on R^+ which implies that $T_{(\cdot)} f$ is Lebesgue integrable on every finite interval of R^+ . Hence, for every finite interval $[\beta, \gamma]$ there exists a sequence $\{f_n\}$ of L_1 valued Lebesgue integrable simple functions on $[\beta, \gamma]$ with the following properties:

- (i) $\lim_{n \rightarrow \infty} \int_{[\beta, \gamma]} \|f_n(t) - f_{m'}(t)\| m(dt) = 0$ and
- (ii) for every $\epsilon > 0$

$$\lim_{n \rightarrow \infty} m\{t \mid t \in [\beta, \gamma], \quad \|T_t f - f_n(t)\| \geq \epsilon\} = 0$$

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and we define

$$\int_{\beta}^{\gamma} T_{if} m(dt) = \lim_{n \rightarrow \infty} \int_{\beta}^{\gamma} f_n(t) m(dt).$$

To give meaning to $\limsup_{\alpha \rightarrow \infty} \int_0^{\alpha} T_{if} m(dt) f \in L_1$, we must give a more explicit definition of $\int_{\beta}^{\alpha} T_{if}(x) m(dt)$ $x \in X$, $0 \leq \beta \leq \gamma$. Consider (P, Σ, ρ) the product measure space of $(R^+, \mathfrak{L}, m)$ and $(X, \mathfrak{F}, \mu)$. Since, by assumption, $T_{\cdot} f$ is Lebesgue integrable for every $f \in L_1$, then for each $f \in L_1$ [6, VII. 7] there exists a real valued ρ measurable function $g(\cdot, \cdot)$ on P , uniquely determined up to ρ measure zero, such that:

- (i) $g(t, \cdot) = T_{if}(\cdot)$ for a.e. $t \in R^+$ and
- (ii) there exists a μ null set $N(f)$ which is independent of t but may depend on f with the property that $g(\cdot, x)$ is Lebesgue integrable on every finite interval of R^+ for every x not in $N(f)$ and $\int_{\beta}^{\gamma} g(t, x) m(dt)$ as a function of x equals $\int_{\beta}^{\gamma} T_{if} m(dt)$ for every x not in $N(f)$.

Hence the meaning of $\int_{\beta}^{\gamma} T_{if}(x) m(dt)$ is obtained from $\int_{\beta}^{\gamma} g(t, x) m(dt)$ for every x not in $N(f)$.

Since $g(\cdot, x)$ is Lebesgue integrable on every finite interval of R^+ for every $x \notin N(f)$ then $\int_0^{\alpha} g(t, x) m(dt)$ is continuous in α on $(0, \infty)$. Let R_r^+ denote the countable set of positive rationals; then from the continuity of $\int_0^{\alpha} g(t, x) m(dt)$ it follows that

$$(i) \quad \inf_{\alpha \geq 0} \sup_{\alpha' \geq \alpha} \int_0^{\alpha'} g(t, x) m(dt) = \inf_{r \in R_r^+} \sup_{\substack{\alpha' \geq r \\ r \in R_r^+}} \int_0^{\alpha'} g(t, x) m(dt)$$

for $x \notin N(f)$.

Meaning is given to $\limsup_{\alpha \rightarrow \infty} \int_0^{\alpha} T_{if} m(dt)$ by

$$\begin{aligned} \left(\limsup_{\alpha \rightarrow \infty} \int_0^{\alpha} T_{if} m(dt) \right) (x) &= \inf_{\alpha \geq 0} \sup_{\alpha' \geq \alpha} \int_0^{\alpha'} g(t, x) m(dt) \\ &= \left(\limsup_{r \rightarrow \infty} \int_0^r T_{if} m(dt) \right) (x) \end{aligned}$$

by (i) for $x \notin N(f)$

where $\limsup$ is taken along the positive reals and positive rationals respectively. Thus $\limsup_{\alpha \rightarrow \infty} \int_0^{\alpha} T_{if} m(dt)$ is a real μ measurable function defined a.e. on X .

We shall use the following abbreviations:

$$S_\beta^\gamma(f) = \int_\beta^\gamma T_t f m(dt), \quad S_\beta^\gamma(f; x) = \int_\beta^\gamma T_t f(x) m(dt),$$

$$\chi_E S_\beta^\gamma(f) = \int_\beta^\gamma \chi_E T_t f m(dt)$$

for $E \in \mathfrak{F}$ where χ_E is the characteristic function of E and $0 \leq \beta \leq \gamma$; if $\beta = 0$ we shall write $S^\gamma(f)$ and $S^\gamma(f; x)$ respectively.

DEFINITION 1. Let $\tau = \{0, t_1, t_2, \dots, t_n\}$ denote a finite partition of R^+ into intervals such that $t_{i-1} < t_i$ for $1 \leq i \leq n = n(\tau)$ and $t_0 = 0$ and define $\tau_i = t_i - t_{i-1}$, $\tau_0 = 0$ for $1 \leq i \leq n$. Denote by $\mathcal{O}$ the class of all such finite partitions τ on R^+ .

DEFINITION 2. For any $\tau \in \mathcal{O}$, $f \in L_1^+$ and $E \in \mathfrak{F}$ we define the following finite sequence of L_1^+ functions $\{a_i(\tau)\}$, $i = 1, 2, \dots, n(\tau)$ by

$$a_0(\tau) = \chi_E f$$

$$a_m(\tau) = \chi_E \left\{ T_{\tau_m} f - \sum_{\nu=1}^m T_{\tau_m - \tau_{m-\nu}} a_{m-\nu}(\tau) \right\}, \quad 1 \leq m \leq n(\tau).$$

It can be shown by induction that

$$a_m(\tau) = \chi_E T_{\tau_m} \chi_E^c T_{\tau_{m-1}} \chi_E^c \cdots \chi_E^c T_{\tau_1} \chi_E^c f, \quad 1 \leq m \leq n(\tau).$$

DEFINITION 3. We define for $E \in \mathfrak{F}$, $f \in L_1^+$ and $\tau \in \mathcal{O}$

$$\Omega_E^\tau(f) = \sum_{\nu=0}^{n(\tau)} \int a_\nu(\tau) d\mu,$$

$$\Omega_E(f) = \sup_{\tau \in \mathcal{O}} \Omega_E^\tau(f)$$

and if $g \in L_1$ then $g = g^+ - g^-$, g^+ and $g^- \in L_1^+$ and we define

$$\Omega_E(g) = \Omega_E(g^+) - \Omega_E(g^-).$$

It can be seen from the definitions that $\Omega_E(\alpha f) = \alpha \Omega_E(f)$ for any $\alpha \geq 0$, $\Omega_E(f) \leq \|f\|$ for any $f \in L^+$, $\Omega_E(f) = 0$ if and only if $S^\infty(f) = 0$ a.e. on E and if the elements of τ are contained in $\tau' (\tau \subset \tau')$ and $f \in L_1^+$ then $\Omega_E^\tau(f) \leq \Omega_E^{\tau'}(f)$.

A maximal ergodic theorem. The following theorem is an extension of the maximal ergodic theorem in [1] to the continuous case. To prove this theorem we shall use discrete approximations.

THEOREM 1. *Let E be any subset of $\mathcal{F}$ and f be any L_1 function. Then $\limsup_{\alpha \rightarrow \infty} S^\alpha(f) > 0$ a.e. on E implies that $\Omega_E(f) \geq 0$ or equivalently $\Omega_E(f^+) \geq \Omega_E(f^-)$ where $f = f^+ - f^-$, f^+ and $f^- \in L_1^+$.*

For this proof we need the following lemmas:

LEMMA 1. *For any $f \in L_1^+$ and $\epsilon > 0$ there exists a rational $r > 0$ and a positive integer n such that*

$$|\Omega_E(f) - \Omega_E^\tau(f)| < \epsilon \quad \text{where} \quad \tau = \{0, r, 2r, \dots, nr\}.$$

PROOF. Choose $\epsilon > 0$; then there exists a partition $\tau^0 \in \mathcal{P}$, $\tau^0 = \{0, t_1, t_2, \dots, t_n\}$ such that

$$|\Omega_E(f) - \Omega_E^{\tau^0}(f)| < \epsilon/2.$$

We now construct a partition $\tau^N = \{0, r_1, r_2, \dots, r_N\}$ where $r_1, r_2, \dots, r_n$ are positive rationals which will be determined by induction and τ^N has the property that $|\Omega_{E_1}^{\tau^0}(f) - \Omega_E^{\tau^N}(f)| < \epsilon/2$. Let τ^m , $1 \leq m \leq N$, denote the partition $\{0, r_1, r_2, \dots, r_{m-1}, r_m, t_{m+1}, \dots, t_N\}$.

For $t, t_1 \leq t < t_2$ we have the following inequalities:

$$\begin{aligned} \text{(i)} \quad & \left| \int_E T_{t_1} \chi_{E^c} f d\mu - \int_E T_t \chi_{E^c} f d\mu \right| \leq \|T_{t_1} \chi_{E^c} f - T_t \chi_{E^c} f\| \\ \text{(ii)} \quad & \left| \int_E T_{t_2-t_1} \chi_{E^c} T_{t_1} \chi_{E^c} f d\mu - \int_E T_{t_2-t} \chi_{E^c} T_t \chi_{E^c} f d\mu \right| \\ & \leq \|T_{t_2-t}(T_{t-t_1} \chi_{E^c} T_{t_1} \chi_{E^c} f - \chi_{E^c} T_{t-t_1} T_{t_1} \chi_{E^c} f)\| \\ & \leq \|T_{t-t_1} \chi_{E^c} T_{t_1} \chi_{E^c} f - \chi_{E^c} T_{t-t_1} \chi_{E^c} f\| + \|T_{t_1} \chi_{E^c} f - T_{t-t_1}(T_{t_1} \chi_{E^c} f)\| \\ \text{(iii)} \quad & \left| \int a_m(\tau^0) d\mu - \int T_{t_m-t_{m-1}} \chi_{E^c} \dots \chi_{E^c} T_{t_2-t} \chi_{E^c} T_t \chi_{E^c} f d\mu \right| \\ & \leq \|T_{t_2-t} \chi_{E^c} T_{t_1} \chi_{E^c} f - T_{t_2-t} \chi_{E^c} T_t \chi_{E^c} f\| \quad \text{for } 3 \leq m \leq N. \end{aligned}$$

By (i), (ii) and (iii) and the continuity of $T_{(\cdot)}f$ in L_1 norm on $[0, \infty)$ we can choose a rational r_1 , $t_1 \leq r_1 < t_2$ such that the right-hand sides of the inequalities (i), (ii) (and hence (iii)) are each less than or equal $\epsilon/2N$.

Define $\tau^1 = \{0, r_1, t_2, \dots, t_N\}$. Then by our choice of r_1 we have

$$|\Omega_E^{\tau^0}(f) - \Omega_E^{\tau^1}(f)| \leq \sum_{\nu=1}^N |\int a_\nu(\tau^0) d\mu - \int a_\nu(\tau^1) d\mu| < \epsilon/2N.$$

In a similar fashion we can find rationals $r_2, r_3, \dots, r_N$, $t_i \leq r_i < t_{i+1}$, $i = 2, 3, \dots, N-1$ and $r_N \geq t_N$, such that the corresponding partitions $\tau^2, \tau^3, \dots, \tau^N$ have the property that

$$|\Omega_E^{\tau^{i-1}}(f) - \Omega_E^{\tau^i}(f)| < \epsilon/2N \quad i = 2, 3, \dots, N.$$

Hence

$$|\Omega_E(f) - \Omega_E^{\tau^N}(f)| \leq |\Omega_E(f) - \Omega_E^{\tau^0}(f)| + \sum_{\nu=1}^N |\Omega_E^{\tau^{\nu-1}}(f) - \Omega_E^{\tau^\nu}(f)| < \epsilon.$$

Let τ denote the reciprocal of the least common denominator of $\{r_1, r_2, \dots, r_N\}$ then $r_i = m_i \tau$ where m_i is some positive integer for $i=1, 2, \dots, N$. Putting $n = \sum_{i=1}^N m_i$, we define the partition $\tau = \{0, \tau, 2\tau, \dots, n\tau\}$. Since $f \in L_1^+$ and $\tau^N \subset \tau$ then

$$0 \leq \Omega_E^{\tau^N}(f) \leq \Omega_E^\tau(f) \leq \Omega_E(f)$$

which implies that $|\Omega_E(f) - \Omega_E^\tau(f)| < \epsilon$.

Note that since T is a strongly continuous semigroup of positive contractions then for $A \in \mathfrak{F}$ and $f \in L_1$, the function $\chi_A T_{(\cdot)} f$ is continuous on $[0, \infty)$ with respect to L_1 norm. Hence we can approximate the function $\chi_A S^\alpha(f)$ in L_1 norm by finite sums of the form $\sum_{i=1}^N \chi_A T_{t_i} f \Delta t_i$ for $\alpha > 0$.

LEMMA 2. Let $(X, \mathfrak{F}, \mu)$ be a σ -finite measure space and $T = \{T_t: t \geq 0\}$ a strongly continuous semigroup of positive contractions. Consider a set $E \in \mathfrak{F}$ and $f \in L_1$ where $f = f^+ - f^-$ ($f^+, f^- \in L_1^+$) and $\chi_E f^- = f^-$. If $\limsup_{\alpha \rightarrow \infty} S^\alpha(f) > 0$ a.e. on E then $\Omega_E(f) \geq 0$.

PROOF. Choose $\epsilon > 0$. The continuity of $S^\alpha(f; x)$, $x \notin N(f)$ and $\limsup_{\alpha \rightarrow \infty} S^\alpha(f) > 0$ a.e. on E imply that for a.e. x in E there exists a rational $r = r(x)$ such that $S^r(f; x) > 0$. Consider the indexed positive rationals $r_1, r_2, \dots$ and define

$$E_i = \{x \mid S^{r_i}(f; x) > 0, x \in E\}.$$

It follows that $E_i \in \mathfrak{F}$ for $i=1, 2, \dots$ and $\mu(\bigcup_{i=1}^\infty E_i) = \mu(E)$.

We choose i sufficiently large, say $i=k$, such that $\int_{E - \bigcup_{i=1}^k E_i} f^- d\mu < \epsilon/4$ (i).

By the construction of E_i , $i=1, 2, \dots, k$, $S^{r_i}(f) > 0$ on E_i hence there exists a measurable subset C_i of E_i and a $\gamma_i > 0$ such that $\chi_{C_i} S^{r_i}(f) \geq \gamma_i$ and $\int_{E_i - C_i} f^- d\mu < \epsilon/4k$. Thus

$$(ii) \quad \bigcup_{i=1}^k E_i = \bigcup_{i=1}^k C_i \cup \bigcup_{i=1}^k (E_i - C_i)$$

and

$$(iii) \quad \int_{\bigcup_{i=1}^k (E_i - C_i)} f^- d\mu \leq \sum_{i=1}^k \int_{E_i - C_i} f^- d\mu < \epsilon/4.$$

For each C_i , $i = 1, 2, \dots, k$ define

$$B_i^n = \left\{ x \in C_i \mid \frac{r_i}{n} \sum_{j=0}^{n-1} \chi_{C_i} T_{r_i/n}^j f(x) \leq 0 \right\} \quad \text{and}$$

$$A_i^n = C_i - B_i^n = \left\{ x \in C_i \mid \frac{r_i}{n} \sum_{j=0}^{n-1} \chi_{C_i} T_{r_i/n}^j f(x) > 0 \right\}.$$

Clearly $B_i^n \in \mathfrak{F}$ for all $n > 0$. Since

$$\lim_{n \rightarrow \infty} \left\| \chi_{C_i} S^{r_i}(f) - \frac{r_i}{n} \sum_{j=0}^{n-1} \chi_{C_i} T_{r_i/n}^j f \right\| = 0,$$

$$0 \leq \|\chi_{B_i^n} S^{r_i}(f)\| \leq \left\| \chi_{C_i} S^{r_i}(f) - \frac{r_i}{n} \sum_{j=0}^{n-1} \chi_{C_i} T_{r_i/n}^j f \right\| \quad \text{for } n > 0$$

and

$$0 \leq \gamma_i \mu(B_i^n) \leq \int_{B_i^n} S^{r_i}(f) d\mu = \|\chi_{B_i^n} S^{r_i}(f)\| \quad \text{for all } n > 0,$$

then for any $\delta > 0$ there exists an integer $M_i = M_i(\delta, i)$ with the property that $\mu(B_i^{M_i}) \leq \delta$, $i = 1, 2, \dots, k$. Since, as a set function, $\int f^- d\mu$ is absolutely continuous with respect to μ then for $\epsilon/4k > 0$ there exists a $\delta = \delta(\epsilon/4k) > 0$ such that if $F \in \mathfrak{F}$ and $\mu(F) \leq \delta$ then $\int_F f^- d\mu \leq \epsilon/4k$. Hence for each i , $i = 1, 2, \dots, k$ there exists an integer $M_i = M_i(\delta, i)$ such that

$$\int_{B_i^{M_i}} f^- d\mu \leq \epsilon/4k, \quad i = 1, 2, \dots, k.$$

Thus

$$(iv) \quad \bigcup_{i=1}^k C_i = \bigcup_{i=1}^k A_i^{M_i} \cup \bigcup_{i=1}^k B_i^{M_i}$$

and

$$(v) \quad \int_{\bigcup_{i=1}^k B_i^{M_i}} f^- d\mu \leq \sum_{i=1}^k \int_{B_i^{M_i}} f^- d\mu \leq \epsilon/4.$$

By putting $A = \bigcup_{i=1}^k A_i^{M_i}$ and using (i), (ii), (iii), (iv), (v) we have

$$\int_E f^- du < \int_A f^- d\mu + 3\epsilon/4.$$

Let r be the reciprocal of the least common denominator of $\{r_i/M, i=1, 2, \dots, k\}$ where $M = \max\{M_i, i=1, 2, \dots, k\}$ then by the construction of A and r it follows that

$$\sup_n \sum_{j=0}^n \chi_A T_{rj}^j f > 0 \quad \text{a.e. on } A.$$

This together with the first part of the proof of Theorem 1 [2] implies that

$$\Omega_A^r(f^+) \geq \int_A f^- d\mu$$

where

$$\Omega_A^r(f^+) = \sum_{v=0}^{\infty} \int a_v d\mu \quad a_0 = \chi_A f \quad \text{and}$$

$$a_n = \chi_A (T_r \chi_A c)^n f \quad \text{for } n \geq 1.$$

There exists an integer N such that for the partition $\tau = \{0, r, 2r, \dots, Nr\}$,

$$\Omega_A^{\tau}(f^+) > \Omega_A^r(f^+) - \epsilon/4.$$

Hence

$$\begin{aligned} \Omega_E(f^+) &\geq \Omega_E^{\tau}(f^+) \geq \Omega_A^{\tau}(f^+) > \Omega_A^r(f^+) - \epsilon/4 \geq \int_A f^- d\mu - \epsilon/4 \\ &> \int_E f^- d\mu - \epsilon = \Omega_E(f^-) - \epsilon. \end{aligned}$$

Therefore, $\Omega_E(f^+) - \Omega_E(f^-) > -\epsilon$ for any $\epsilon > 0$ which implies $\Omega_E(f^+) \geq \Omega_E(f^-)$ which completes the proof.

LEMMA 3. Let f and $g \in L_1^+$, then for any $\beta > 0$

$$\lim_{\alpha \rightarrow \infty} \frac{S_{\alpha}^{\alpha+\beta}(g)}{S^{\alpha}(f)} = 0 \quad \text{a.e. on } \{x \in X \mid S^{\infty}(f; x) > 0\}.$$

PROOF. Choose $\beta > 0$. For any real number $\alpha \geq \beta$ there exists an integer $n = n(\alpha, \beta)$ with the following properties: $n\beta \leq \alpha$ and $\alpha = \alpha_0 + \beta$ where $\alpha_0 = \alpha_0(\alpha, \beta)$ and $0 \leq \alpha_0 < \beta$.

Hence $S_{\alpha}^{\alpha+\beta}(g) = T_{\beta}^n S_{\alpha_0}^{\alpha_0+\beta}(g) \leq T_{\beta}^n S^{2\beta}(g)$ a.e. on X and $S^{\alpha}(f) \geq S^{n\beta}(f) = \sum_{v=0}^n T_{\beta}^v S^{\beta}(f)$ a.e. on X . This yields

$$0 \leq \frac{S^{\alpha+\beta}(g)}{S^\alpha(f)} \leq \frac{T_\beta^n S^{2\beta}(g)}{\sum_{\nu=0}^n T_\beta^\nu S^\beta(f)} \quad \text{a.e. on the support of } \sum_{\nu=0}^n T_\beta^\nu S^\beta(f).$$

Since

$$\lim_{n \rightarrow \infty} \frac{T_\beta^n S^{2\beta}(g)}{\sum_{\nu=0}^n T_\beta^\nu S^\beta(f)} = 0$$

a.e. on

$$\left\{ x \in X \mid \left(\sum_{\nu=0}^{\infty} T_\beta^\nu S^\beta(f) \right)(x) > 0 \right\} = \{ x \in X \mid S^\infty(f; x) > 0 \} = A$$

then

$$\lim_{\alpha \rightarrow \infty} \frac{S^{\alpha+\beta}(g)}{S^\alpha(f)} = 0 \quad \text{a.e. on } A$$

which completes the proof.

LEMMA 4. *Given f and $g \in L_1^+$, then for any real number $\beta > 0$ and integer $n \geq 0$ define*

$$\tilde{g} = \chi_E g + \chi_E T_\beta \chi_E g + \chi_E T_\beta \chi_E T_\beta \chi_E g + \cdots + \chi_E (T_\beta \chi_E)^n g.$$

If, for any $0 < \lambda < 1$ and $S^\infty(g) > 0$ a.e. on E ,

$$\limsup_{\alpha \rightarrow \infty} S^\alpha(f - g) > 0 \quad \text{a.e. on } E \text{ then}$$

$$\limsup_{\alpha \rightarrow \infty} S^\alpha(f - \lambda \tilde{g}) > 0 \quad \text{a.e. on } E.$$

PROOF. Choose a real number $\beta > 0$ and an integer $n \geq 0$ and let $a_0 = \chi_E g$ and $a_\nu = \chi_E (T_\beta \chi_E)^\nu g = \chi_E (T_\beta^\nu g - \sum_{j=1}^\nu T_\beta^j a_{\nu-j})$ for $\nu = 1, 2, \dots, n$; define $\tilde{g} = \sum_{\nu=0}^n a_\nu$.

For any $\alpha > 0$ we have

$$(i) \quad S^\alpha(f - \lambda \tilde{g}) = S^\alpha(f) - \lambda \left[\sum_{\nu=0}^{n-1} S^\alpha(a_\nu) - S^\alpha(a_n) \right].$$

Since we are restricting ourselves to the set E nothing is lost by writing

$$a_n = T_\beta^n g - \sum_{j=1}^n T_\beta^j a_{n-j} = T_\beta \left[T_\beta^{n-1} g - \sum_{j=1}^{n-1} T_\beta^j a_{n-1-j} - a_{n-1} \right]$$

$$\cdot S^\alpha(a_n) = S_\beta^{\alpha+\beta} \left(T_\beta^{n-1} g - \sum_{j=1}^{n-1} T_\beta^j a_{n-1-j} - a_{n-1} \right)$$

$$\leq S^{\alpha+\beta} \left(T_\beta^{n-1} g - \sum_{j=1}^{n-1} T_\beta^j a_{n-1-j} - a_{n-1} \right).$$

Substitution in (i) and simple calculations yield

$$S^\alpha(f - \lambda \tilde{g}) \geq S^\alpha(f) - \lambda \left[\sum_{\nu=0}^{n-2} S^\alpha(a_\nu) - S^{\alpha+\beta} \left(T_\beta^{n-1} g - \sum_{j=1}^{n-1} T_\beta^j a_{n-1-j} \right) \right].$$

By repeating this process for $a_{n-1}, a_{n-2}, \dots, a_1$, we have

$$S^\alpha(f - \lambda \tilde{g}) \geq S^\alpha(f) - \lambda S^\alpha(g) - \lambda S_\alpha^{\alpha+n\beta}(g) \quad \text{for every } \alpha \geq 0 \text{ a.e. on } E,$$

which implies that

$$\limsup_{\alpha \rightarrow \infty} S^\alpha(f - \lambda \tilde{g}) \geq \limsup_{\alpha \rightarrow \infty} [S^\alpha(f) - \lambda S^\alpha(g) - \lambda S_\alpha^{\alpha+n\beta}(g)].$$

Since

$$\lim_{\alpha \rightarrow \infty} \lambda \frac{S_\alpha^{\alpha+n\beta}(g)}{S^\alpha(g)} = 0 \quad \text{a.e. on } \{x \mid S^\infty(g; x) > 0\}$$

by Lemma 3 and

$$\limsup_{\alpha \rightarrow \infty} S^\alpha(f - g) > 0 \quad \text{a.e. on } E$$

then it follows that

$$\limsup_{\alpha \rightarrow \infty} [S^\alpha(f) - \lambda S^\alpha(g) - \lambda S_\alpha^{\alpha+\beta}(g)] > 0$$

a.e. on $E' = E \cap \{x \mid S^\infty(g; x) > 0\}$ and that $E - E'$ is a set of μ measure zero. Thus $\limsup_{\alpha \rightarrow \infty} S^\alpha(f - \lambda \tilde{g}) > 0$ a.e. on E which completes the proof.

PROOF OF THEOREM 1. Choose any $\epsilon > 0$. By Lemma 1 there exists a rational $r > 0$ and an integer n such that on $E' = E \cap \{x \mid S^\infty(f^-; x) > 0\}$

$$|\Omega_{E'}(f^-) - \Omega_{E'}^r(f^-)| < \epsilon \quad \text{where } r = \{0, r, 2r, \dots, nr\}$$

and $\Omega_{E'}^r(f^-) = \sum_{\nu=0}^n \int a_\nu(\tau) d\mu$ where $a_0(\tau) = \chi_{E'} f^-$ and $a_\nu = \chi_{E'}(T_r \chi_{E'}^{\nu-1} f^-)$ for $1 \leq \nu \leq n$.

Define $g = \sum_{\nu=0}^n a_\nu(\tau)$ then it follows from Lemmas 4 and 2 respectively that, for any $0 < \lambda < 1$, $\limsup_{\alpha \rightarrow \infty} S^\alpha(f^+ - \lambda g) > 0$ a.e. on E' and $\Omega_{E'}(f^+) \geq \lambda \Omega_{E'}(g)$ which yields that $\Omega_{E'}(f) \geq \Omega_{E'}(g)$. Hence $\Omega_{E'}(f^+) \geq \Omega_{E'}(g) = \Omega_{E'}^r(f^-) > \Omega_{E'}(f^-) - \epsilon$ which yields $\Omega_{E'}(f^+) \geq \Omega_{E'}(f^-)$. Since $S^\infty(f^-) = 0$ a.e. on $E - E'$ then $\Omega_{E-E'}(f^-) = 0$. Hence

$$\Omega_E(f^+) \geq \Omega_{E'}(f^+) \geq \Omega_{E'}(f^-) + \Omega_{E-E'}(f^-) \geq \Omega_E(f^-)$$

which completes the proof.

A ratio ergodic theorem. The following theorem which extends the ratio ergodic theorem of Chacon and Ornstein to the continuous case is a direct consequence of Theorem 1 and its proof is identical to the proof of Theorem 2 in [2] so only an outline of the proof will be given.

THEOREM 2. *Let $f \in L_1^+$, $g \in L_1$ and $E \in \mathfrak{F}$, then $\lim_{\alpha \rightarrow \infty} S^\alpha(g)/S^\alpha(f)$ exists (and is finite) a.e. on $E' = E \cap \{x \in X \mid S^\infty(f; x) > 0\}$.*

PROOF. If $\limsup_{\alpha \rightarrow \infty} S^\alpha(g)/S^\alpha(f)$ is infinite a.e. on E' then by Theorem 1,

$\Omega_B(g) \geq \gamma \Omega_B(f)$ for some nonnegligible subset B of E and for all $\gamma \geq 0$ which is a contradiction since $\Omega_B(g) < \infty$ and $\Omega_B(f) > 0$.

If $\lim_{\alpha \rightarrow \infty} S^\alpha(g)/S^\alpha(f)$ does not exist a.e. on E' then there exists a nonnegligible subset G of E' , two numbers $\gamma < \beta$ and a nonnegligible subset G' of G with property that

$$\liminf_{\alpha \rightarrow \infty} \frac{S^\alpha(g)}{S^\alpha(f)} \leq \gamma < \beta \leq \limsup_{\alpha \rightarrow \infty} \frac{S^\alpha(g)}{S^\alpha(f)} \quad \text{a.e. on } G'.$$

This implies, by Theorem 1, that $\gamma \Omega_{G'}(f) \geq \Omega_{G'}(g)$ and $\Omega_{G'}(g) \geq \beta \Omega_{G'}(f)$ which is a contradiction.

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