$B(H)$ IS VERY NONCOMMUTATIVE

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For a W -* algebra the following two properties are partial forms of commutativity.

DEFINITION 1. A W -* algebra $\mathfrak{A}$ has *property L* if there are unitary operators $U_n \in \mathfrak{A}$ such that $U_n \rightarrow 0$ weakly and such that for every $A \in \mathfrak{A}$, $U_n A U_n^* \rightarrow A$ strongly.

DEFINITION 2. A W -* algebra $\mathfrak{A}$ has *property A.A.* if there are *-automorphisms ϕ_n of $\mathfrak{A}$ such that for every pair $S, T \in \mathfrak{A}$, $[S, \phi_n(T)] \rightarrow 0$ strongly, where $[S, T] = ST - TS$.

Definition 1 is due to Pukánszky [3] and Definition 2 is essentially due to Sakai [4].

Let H denote a separable Hilbert space and let $B(H)$ denote the algebra of all bounded operators on H . It is our aim to prove that $B(H)$ has neither property A.A. nor property L.

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LEMMA 3. Let V_n be a sequence of unitary operators such that $V_n^* A V_n \rightarrow A$ strongly for each $A \in B(H)$. Then there is a subsequence (again called V_n) and a scalar λ such that $|\lambda| = 1$ and $V_n \rightarrow \lambda I$ strongly.

PROOF. Clearly $A V_n - V_n A \rightarrow 0$ strongly. By the weak compactness of the unit ball of operators we may assume that V_n (or a subsequence) converges weakly to some V . It follows readily that $A V = V A$ for all $A \in B(H)$ and hence that $V = \lambda I$, $|\lambda| \leq 1$.

Let $\{x_n\}$ be an orthonormal basis for H and define an operator A such that $Ax_1 = x_1$, $Ax_j = 0$, $j > 1$. A simple calculation shows that

$$((V_n^* A V_n - A)x_1, x_1) = |(V_n x_1, x_1)|^2 - 1 \rightarrow 0.$$

Since clearly $(V_n x_1, x_1) \rightarrow \lambda$, we see that $|\lambda|^2 = 1$. Thus $|\lambda| = 1$. Since the V_n are unitary and converge weakly to the unitary λI , it follows that $V_n \rightarrow \lambda I$ strongly. Q.E.D.

COROLLARY 4. $B(H)$ does not have property L.

PROOF. Use Lemma 3 putting $V_n = U_n^*$. Q.E.D.

THEOREM 5. $B(H)$ does not have property A.A.

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PROOF. Since it is well known that every $*$ -automorphism of $B(H)$ is inner (see [1]) we may assume that $\phi_n(A) = U_n A U_n^*$ for some unitary U_n . Suppose therefore that $B(H)$ has property A.A. and that $\{U_n\}$ is the corresponding sequence of unitaries demonstrating the property. It follows that for each pair $A, B \in B(H)$ we have $U_n A U_n^* B - B U_n A U_n^* \rightarrow 0$ strongly.

Let E denote the orthogonal projection on the span of $\{x_1, x_3, \dots\}$, and let $F = I - E$. Then there is a unitary V such that $E = V F V^*$, and there are operators P, Q, R , and S such that $I = S E R = P F Q$. Let $\{V_m\}$ be a countable set of unitaries such that E, F, P, Q, R, S , and V are linear combinations of the V_m and such that $\{V_m\}$ is closed under multiplication and taking adjoints. Let $\mathcal{Q}$ be the involutive algebra generated by the $\{V_m\}$. We may assume that the U_n are such that the strong limit of the sequence $U_n V_m U_n^* = \lambda_m I$ exists for each V_m , by Lemma 3 and the Cantor diagonal process. Then it is readily verified that if $A \in \mathcal{Q}$, $A = \sum_{m=1}^N a_m V_m$, we may define $\phi: \mathcal{Q} \rightarrow \mathbb{C}$ by $\phi(A) = \lambda$ where strong limit $U_n A U_n^* = \lambda I$. It is easy to check that ϕ is linear, multiplicative, and $\phi(I) = 1$. Hence $\phi(E) = 0$ or $\phi(E) = 1$. But $\phi(F) = \phi(V)\phi(E)\phi(V^*)$, and therefore $\phi(F) = \phi(E)$. Therefore $\phi(I) = \phi(E) + \phi(F)$, so that $\phi(I) = 0$ or $\phi(I) = 2$, both contradictions. It follows that $B(H)$ does not have property A.A. Q.E.D.

This proof is essentially the proof of Theorem 4.1 in [2].

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