ON CONTINUOUSLY SEMIMETRIZABLE AND STRATIFIABLE SPACES

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1. Introduction. Recently Professor J. Nagata asked us a question which we will phrase as follows: Is it true that a space X is a Nagata space (i.e. stratifiable² and first countable) if and only if X is i -continuously semimetrizable³ (see Definition 2.1)? Unfortunately, it turns out that neither implication is valid even for regular cosmic⁴ (hence paracompact) spaces. The situation is not improved by replacing " i -continuously" by "continuously" in the above question. Nevertheless, it should not be forgotten that every Nagata space is semimetrizable (see Theorem 6.2 of Ceder [2]).

Throughout we use the terminology of Kelley [6].

2. Theorems and proofs. Before answering Nagata's question we establish a few preliminaries.

DEFINITION 2.1. Let $f: X \times Y \rightarrow Z$ be a function, where X , Y and Z are topological spaces. Then f is said to be i -continuous ($i = 1, 2$) if f is continuous with respect to the i th variable.

LEMMA 2.2. *Let X be a semimetrizable space and d be a semimetric on X which generates the topology on X . Then d is 1-continuous if and only if, for each $(p, p') \in X \times X$ and sequence $\{x_n\}_{n=1}^{\infty}$ converging to p , we get that $\lim_n d(x_n, p') = d(p, p')$.*

PROOF. Straightforward, since X is clearly first countable.

EXAMPLE 2.3. There exists a cosmic (see footnote 4) Nagata space X which is not 1-continuously semimetrizable.

PROOF. We actually use a space first constructed by Heath [4]. Let X be the set of all points of the plane with a base for a topology τ

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² A T_1 -space X is stratifiable if to each open $U \subset X$ one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of X such that, for all n , $U_n \subset U$, $U_n \subset V_n$ whenever $U \subset V$, and $\bigcup_{n=1}^{\infty} U_n = U$. The correspondence $U \rightarrow \{U_n\}_{n=1}^{\infty}$ is called a stratification of X . Stratifiable spaces are the same as the M_3 -spaces of Ceder [2].

³ A T_1 -space X is semimetrizable if there exists a real-valued function d on $X \times X$ such that (a) $d(x, y) = d(y, x)$, (b) $d(x, y) = 0$ if and only if $x = y$, and (c) a point $p \in X$ is in the closure of a subset B of X if and only if $\inf\{d(p, b) \mid b \in B\} = 0$ (i.e. d generates the topology of X). The function d is called a semimetric. X is i -continuously semimetrizable if d is i -continuous.

⁴ A cosmic space is any T_1 -regular space which is the continuous image of a separable metric space.

consisting of (1) all open discs and (2) all "butterfly regions" centered on irrational points of the x -axis (a "butterfly region" centered at the point x is $\{y \in X \mid |x-y| + a(x, y) < c\}$, where $|x-y|$ is the cartesian distance between x and y and $a(x, y)$ is the smallest nonnegative angle, expressed in radians, formed by the x -axis and the line joining x and y). For convenience, we let $N(x, c) = \{y \in X \mid |x-y| + a(x, y) < c\}$ and $D((x, y), r)$ denote the open disc centered at (x, y) and with radius r .

It is easily seen that (X, τ) is stratifiable: For each open $U \subset X$ and positive integer n , let $U_n = U'_n \cup U''_n$, where

$$U'_n = \bigcup \left\{ D\left((x, y), \frac{1}{2n}\right) \mid D\left((x, y), \frac{1}{n}\right) \subset U \right\},$$

$$U''_n = \bigcup \left\{ N\left(x, \frac{1}{2n}\right) \mid N\left(x, \frac{1}{n}\right) \subset U \right\}.$$

A simple argument shows that $(U''_n)^- \subset U$. Obviously, $(U'_n)^- \subset U$, $\bigcup_{n=1}^{\infty} U_n = U$ and $U_n \subset V_n$ whenever $U \subset V$. Consequently (X, τ) is stratifiable (see footnote 2). Since X is first countable then X is a Nagata space.

However, Heath [4] proved that there exists no semimetric on X , which generates the topology of X , under which all spheres are open. As pointed out by Sims in [7], it follows that X is not 1-continuously semimetrizable. (This is also easily seen by the use of Lemma 2.2.)

It is easily seen that X is cosmic, since the subspaces $A = \{(x, y) \in X \mid y = 0\}$ and $Y = X - A$ of X are separable metrizable spaces and thus X is the continuous image of the disjoint topological union $A \vee Y$ of A and Y .

EXAMPLE 2.4. There exists a cosmic (hence paracompact) 1-continuously semimetrizable space which is not stratifiable.

PROOF. Let X be the set of all points (x, y) of the plane such that

(1) $y = 0$ and $x, \sqrt{2/n} - x, \sqrt{2/n} + x$ are irrational for each positive integer n , or

(2) x is rational and $y = \sqrt{2/n}$ for some positive integer n .

A base for a topology τ on X consists of all sets $B((x, y), n) = \{(x, y)\} \cup \{(w, z) \in X \mid |w-x| < 1/n \text{ and } 0 \leq z-y < |w-x|\}$, (i.e. $B((x, y), n)$ is a "butterfly region centered at (x, y) with radius $1/n$ and vertex angle $\pi/4$ ") for any $(x, y) \in X$ and positive integer n .

Actually a "small" neighborhood of a point $(x, \sqrt{2/n}) \in X$ is just an open interval of rational numbers in the horizontal line passing through $(x, \sqrt{2/n})$ and containing $(x, \sqrt{2/n})$. Also the only boundary points of a neighborhood $B((x, 0), n)$ are $(x+1/n, 0)$ and $(x-1/n, 0)$,

since the hypotenuses of the "wings" of the butterfly $B((x, 0), n)$ contain no points $(w, \sqrt{2}/n) \in X$ because of (1). Consequently one immediately sees that (X, τ) is a regular space.

Proceeding as in the last paragraph of the proof of Example 2.3 we can thus easily see that X is a cosmic space.

Now we define a semimetric d on $X \times X$ as follows:

$$\begin{aligned} d(x, y) &= |x - y| && \text{if } a(x, y) \leq \pi/4, \\ &= |x - y| + a(x, y) && \text{if } a(x, y) > \pi/4 \end{aligned}$$

where $a(x, y)$ denotes the smallest positive angle (expressed in radians) between the x -axis and the line joining x and y (we let $a(x, x) = 0$).

It is easily seen that d is a semimetric on $X \times X$ which generates the topology on X . Furthermore one can easily show that d is 1-continuous by the use of Lemma 2.2 and the crucial fact (as observed by the referee) that $a(x, y) \neq \pi/4$ for all $x, y \in X$.

To complete the proof, we need only show that X is not stratifiable. (We essentially use an argument which first appears in Heath [5]. However there are enough necessary changes which seem to warrant a complete proof.) Suppose that X is stratifiable and $U \rightarrow \{U_n\}_{n=1}^\infty$ is a stratification of X . Let $X_{kj} = \{(x, 0) \in X \mid B((x, 0), k) \subset U_j \subset U \subset B((x, 0), 1) \text{ for some open } U \subset X\}$ for any positive integers k and j . Clearly $\bigcup_{k,j=1}^\infty X_{kj} = \{(x, 0) \mid (x, 0) \in X\}$ and the real line minus the set of all $(x, 0) \in X$ is a countable set (it consists of rationals and one more irrational for each rational and each positive integer n). Therefore the closure of some X_{mn} in the real line contains an open interval $]a, b[$, since the real line is of second category. Pick any $a < r < b$ such that $r + \sqrt{2}/2m$ is rational (this is easily done!). Then r is in the closure of $\{(x, 0) \in X_{mn} \mid x > r\}$ or $\{(x, 0) \in X_{mn} \mid x < r\}$; assume the former. Let $\mathfrak{Q}$ be the set of all (U, U_n) such that

$$B((x, 0), m) \subset U_n \subset U \subset B((x, 0), 1)$$

for some $x \in X_{mn}$ with $r < x < r + \sqrt{2}/m$. Then, letting $y = (r + \sqrt{2}/2m, \sqrt{2}/2m) \in X$, we get that

$$y \in (\bigcup \{U_n \mid (U, U_n) \in \mathfrak{Q}\})^-$$

since $y \in \bigcup \{B((x, 0), m) \mid x \in X_{mn}, r < x < r + \sqrt{2}/m\}$ because $1/m > \sqrt{2}/2m$. However,

$$y \notin \bigcup \{U \mid (U, U_n) \in \mathfrak{Q}\}$$

since $y \notin \bigcup \{B((x, 0), 1) \mid x \in X_{mn}, r < x < r + \sqrt{2}/m\}$, a contradiction

since we clearly have that

$$\begin{aligned} (\cup \{U_n \mid (U, U_n) \in \alpha\})^- &\subset (\cup \{U \mid (U, U_n) \in \alpha\})_n^- \\ &\subset \cup \{U \mid (U, U_n) \in \alpha\}, \end{aligned}$$

by the assumption that X is stratifiable.

Concluding remarks. By the symmetry of a semimetric, obviously a 1-continuous semimetric is 2-continuous and vice versa. Hence the semimetric d of Example 2.4 is *separately*-continuous (i.e. continuous in each variable). However, it is not hard to see that d is not continuous (consider any point $(x, 0) \in X$ and sequences $\{p_n\}_{n=1}^\infty \{q_n\}_{n=1}^\infty$ converging to $(x, 0)$ such that $a(p_n, q_n) > \pi/4$ for each n —this can be easily done by choosing all p_n and q_n in the left wing of the butterfly $B((x, 0), n)$ with p_n below q_n).

One might wonder then: Is every continuously semimetrizable space a Nagata space? Cook has answered that question. He has shown in [3, Theorems 7 and 8] that every continuously semimetrizable space is developable and that there exists a continuously semimetrizable space that is not normal and therefore necessarily not a Nagata space (or stratifiable).

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