RETRACTING DIFFEOMORPHISMS OF n -SPHERES

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1. Introduction. Connectedness properties of various spaces of orientation preserving homeomorphisms of the n -sphere S^n onto itself have been studied for some time. If K_n , K'_n , K''_n are the spaces of orientation preserving homeomorphisms with uniform convergence topology, C^∞ orientation preserving homeomorphisms (not requiring differentiable inverses) with C^k topology and C^∞ orientation preserving diffeomorphisms with C^k topology respectively then the following results are known: the rotation group of S^2 is a strong deformation retract of K_2 (see [3]), the rotation group of S^2 is a strong deformation retract of K''_2 (see [7]) and K'_n is arcwise connected (see [6]). There are maps in K''_6 not differentiably isotopic to the identity (see [4]). K_2 and K_3 are locally arcwise connected (see [5]), and if K'_3 and the rotation group on S^3 are homotopically equivalent then K_3 and the rotation group on S^3 are also (see [2]).

If one replaces the C^k topology in K''_n by its coarser pointwise counterpart where arcs are more plentiful, then the space has very nice connectedness properties. In particular the purpose of this paper is to prove the following:

THEOREM. *The space of all C^∞ orientation preserving diffeomorphisms of S^n onto itself under the pointwise C^k topology has as a strong deformation retract the rotation group of S^n .*

Necessary mappings are constructed.

2. Pointwise C^k topology and the spaces H''''_n . Let D_n be the closed unit disk in E_n and H''_n be the space of C^∞ diffeomorphisms of D_n onto itself which are the identity in a neighborhood of the boundary under the C^k topology. Let H''''_n be the same set of functions under the pointwise C^k topology. Convergence in H''_n and K''_n requires uniform convergence through the first k derivatives while convergence in H''''_n and K''_n only requires *pointwise convergence through those k derivatives*. A base of neighborhoods of a function f consists of sets of functions whose first k derivatives differs from those of f by less than $\epsilon > 0$ at a finite set of points, ϵ and the finite set varying. In particular the C^k topology is finer than the pointwise C^k topology.

LEMMA. *H''''_n is contractible to the identity map on D_n .*

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PROOF. For $0 \leq t \leq 1$ let $D(t)$ be the n disk given by

$$\{x = (x_1, x_2, \dots, x_n) \in E^n: |x - (-t/2, 0, \dots, 0)| \leq -(3t/4) + 1\}.$$

Let $T(t)$ be the obvious mapping of $D(t)$ onto D_n defined by $T(x_1, \dots, x_n) = (4/(-3t+4))(x_1+t/2, x_2, \dots, x_n)$. Now for $f \in H_n'''$ and for $0 \leq t \leq 1$ define $f_t: D_n \rightarrow D_n$ by:

$$\begin{aligned} f_t(x) &= x & \text{if } x \notin D(t), \\ &= T^{-1}f(Tx) & \text{if } x \in D(t). \end{aligned}$$

The mapping of $H_n''' XI \rightarrow H_n'''$ defined by $\theta_1(f, t) = f_t$ retracts H_n''' onto a subspace of H_n''' such that if h is in the subspace, then h is the identity map in a neighborhood of the origin and of the boundary of D_n .

The Alexander mapping (see [1]) applied to this subspace shows H_n''' is contractible to the identity map. In particular the Alexander mapping $\theta_2: H_n''' XI \rightarrow H_n'''$ is defined by $\theta_2(h, t) = h^t$ where

$$\begin{aligned} h^t(x) &= th(x/t), & 0 < t \leq 1, & \quad (h \text{ extended to be the identity outside } D_n) \\ &= x, & t = 0. \end{aligned}$$

(Note that θ_1 is just a sliding Alexander type map restricted to $\frac{1}{4} \leq t \leq 1$.) Then the map $\theta: H_n''' XI \rightarrow H_n'''$ defined by

$$\begin{aligned} \theta(f, t) &= \theta_1(f, 2t), & 0 \leq t \leq \frac{1}{2}, \\ &= \theta_2(\theta_1(f, 1), -2t + 2), & \frac{1}{2} \leq t \leq 1 \end{aligned}$$

is the required contraction mapping.

θ is continuous in the C^k topology except at $t=1$. Also given $\theta(f, 1)$ and a point x , if $x=0$ then all functions $\theta(g, t)$, $g \in H_n'''$, $0 \leq t \leq \frac{1}{2}$ have all derivatives equal to $\theta(f, 1)$ at x . If $x \neq 0$ then there is a neighborhood N_f of f and an interval $(\zeta, 1]$ such that $\theta(g, t)$ has derivatives equal to $\theta(f, 1)$ at x if $g \in N_f$ and $t \in (\zeta, 1]$.

3. Deforming K_n''' .

THEOREM. *The rotation group of S^n is a strong deformation retract of K_n''' .*

PROOF. The necessary retraction mapping can be constructed from mappings given in paragraphs III and IV of [6].

Let $p = (0, 0, \dots, -1)$ on S^n and $P: \{(x_1, \dots, x_{n+1}) \in S_n: x_{n+1} \leq 0\} \rightarrow E_n$ be the projection map $P(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$. For $f \in K_n'''$ there is a rotation r_f of S^n such that $r_f(p) = p$ and the Jacobian matrix of $Pr_f f(P^{-1}x)$ at $x=0$ is triangular with positive

diagonal elements. Using mappings of [6, paragraph III], the mapping $Pr_{ff}P^{-1}$ can be deformed by a mapping $(Pr_{ff}P^{-1})_t$, $0 \leq t \leq 1$ which at $t=1$ is the identity in a neighborhood N_f of the origin in E_n and agrees with $(Pr_{ff}P^{-1})_0 = Pr_{ff}P^{-1}$ outside a larger neighborhood N . Furthermore the size of N can be fixed over all $f \in K_n'''$ with the deformation continuous with respect to f in K_n''' . We now construct the retraction map ψ which deforms K_n''' onto the rotation group of S_n .

Define $\psi_1(f, t)$ for $(f, t) \in K_n''' \times I$ by

$$\begin{aligned}\psi_1(f, t)(x) &= r_f^{-1}P^{-1}(Pr_{ff}P^{-1})_t(P(x)) && \text{if } P(x) \in N \\ &= f(x) && \text{if } P(x) \notin N.\end{aligned}$$

Then ψ_1 retracts K_n''' onto a subspace J_n''' such that if $g \in J_n'''$ then r_0g is the identity in a neighborhood of p . Retracting J_n''' onto the rotation group is now easy using the lemma. Specific details exactly parallel those given in [6, paragraph IV] or in [7].

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