PATH MAPS AND CONTINUITY¹

ERIC BEDFORD

A function $f: X \rightarrow Y$ will be called a *path map* if it maps the paths of X to paths of Y . That is, if $\sigma: I \rightarrow X$ is a path, then so is $f\sigma: I \rightarrow Y$. Any continuous function is a path map. Conversely, I shall show that with mild restrictions on the space X , all path maps $f: X \rightarrow Y$ are continuous.

A duality consideration yields some more information about the situation. In this case, the object of interest is the following: $g: X \rightarrow Y$ is a *reverse path map* if whenever $\phi: Y \rightarrow I$ is a continuous (real-valued) function, then so is $\phi g: X \rightarrow I$. If certain conditions are placed upon Y , then every reverse path map $g: X \rightarrow Y$ is continuous. The aim of the following theorems is to reveal some of the interplay between paths, continuous functions, and continuous real-valued functions.

THEOREM 1. *If X is locally path connected, Hausdorff and 1st countable, then every path map $f: X \rightarrow Y$ is continuous.*

PROOF. Let $\{x_n\}$ be any sequence in X converging to x_0 . We shall construct a path $\Gamma[0, 1] \rightarrow X$ such that $\Gamma(1/n) = x_n$, $\Gamma(0) = x_0$. Let $U_1 \supseteq U_2 \supseteq \dots$ be a *nb.* basis at x_0 with U_n path-connected. We may assume $x_n \in U_1$, $n = 1, 2, \dots$. For each $n \geq 1$, let j_n be the largest integer such that x_n and x_{n+1} lie in U_{j_n} . Pick a path $\gamma_n: [1/(n+1), 1/n] \rightarrow U_{j_n}$ such that $\gamma_n(1/(n+1)) = x_{n+1}$, $\gamma_n(1/n) = x_n$. Now we define $\Gamma: [0, 1] \rightarrow X$ as:

$$\Gamma(0) = x_0, \quad \Gamma(t) = \gamma_n(t) \quad \text{if } t \in [1/(n+1), 1/n].$$

Clearly $\Gamma: (0, 1] \rightarrow X$ is continuous. To see that Γ is continuous at 0, let V be any *nb.* of x_0 . Then for some integer K , $U_j \subseteq V$ whenever $j \geq K$. Let N be large enough that if $n \geq N$, then $x_n \in U_K$. Such an N exists because $\{x_n\}$ converges to x_0 . Further, this means that if $j \geq N$, then the path γ_j lies inside of U_K . Hence $\Gamma^{-1}([0, 1/N]) \subseteq U_K \subseteq V$, and thus Γ is continuous.

Since f is a path map, $f\Gamma$ will be a path in Y such that $f\Gamma(1/n) = f(x_n)$ and $f\Gamma(0) = f(x_0)$. Thus $f(x_n)$ converges to $f(x_0)$, and f is continuous. Q.E.D.

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THEOREM 2. *If X and Y are locally path connected and Hausdorff and 1st countable, and if $f: X \rightarrow Y$ is any (set theoretic) function, then the following are equivalent:*

- (i) f is a homeomorphism,
- (ii) f^{-1} exists, and both f and f^{-1} are path maps,
- (iii) f is a path map and the induced $\hat{f}$ (i.e., $\hat{f}(\sigma) = f\sigma$) is a set theoretic isomorphism of the sets X^I and Y^I ,
- (iv) f is a path map and the induced $\hat{f}$ is a homeomorphism in the compact-open topology.

PROOF. This is an immediate consequence of Theorem 1. The maps $\hat{f}$ that are induced from path maps $X \xrightarrow{f} Y$ can be described easily.

THEOREM 3. *If $G: X^I \rightarrow Y^I$ is a set-theoretic map, and G is invariant under parametrization (i.e., if $\rho: I \rightarrow I$ is a continuous map, then $G(\sigma)\rho = G(\sigma\rho)$), then there is a path map $f: X \rightarrow Y$ such that $\hat{f} = G$.*

PROOF. If τ_{x_0} is the trivial path at x_0 , then $G(\tau_{x_0})$ is a trivial path in Y . Let $\rho_{t_0}: I \rightarrow I$ be the constant map into t_0 . Then $G(\tau_{x_0}) = G(\tau_{x_0}\rho_{t_0}) = G(\tau_{x_0})\rho_{t_0}$, and hence $G(\tau_{x_0})$ is trivial.

Now we define a map: $f: X \rightarrow Y$ as follows. If $x \in X$, and τ_x is the constant path at x , then there is a $y \in Y$ such that $G(\tau_x) = \tau_y$. We define $f(x) = y$. Thus $G(\tau_x) = \tau_{f(x)}$. It follows that f is a path map with $\hat{f} = G$ if we show that $G(\sigma)(t) = f\sigma(t)$ for all $t \in I$. For each $t_0 \in I$, we define $\rho_{t_0}: I \rightarrow I$ as the constant map into t_0 . Then, since $G(\sigma)\rho = G(\sigma\rho)$, we get $G(\sigma)(t_0) = \{G(\sigma)\rho_{t_0}\}(s) = G(\sigma\rho_{t_0})(s) = G\{\tau_{\sigma(t_0)}\} = f(\sigma(t_0))$. Thus $\hat{f} = G$. Q.E.D.

REMARK. Theorem 3 is valid for arbitrary spaces X and Y . Also, invariance under parametrization is a necessary condition for G to be a $\hat{f}$. Finally, if X is locally path connected, Hausdorff and 1st countable, and if G is invariant under parametrization, then G is continuous (in the $C-O$ topology).

We next consider the action of the functor π^I (i.e., $\pi^I(X) = X^I$) from T (topological spaces) to S (sets). Let $\mathfrak{X}$ be any subclass of T . We define the category $T(\mathfrak{X})$ to contain all topological spaces as objects, but the morphisms of $T(\mathfrak{X})$ are only those continuous maps that emanate from spaces of $\mathfrak{X}$. We also define the category $S(\mathfrak{X})$ whose objects are Z^I , for $Z \in T$. The morphisms of $S(\mathfrak{X})$ are the maps $G: X^I \rightarrow Y^I$ such that $x \in \mathfrak{X}$, and G is invariant under parametrization. If the spaces of $\mathfrak{X}$ are locally path connected, Hausdorff and 1st countable, then π^I from $T(\mathfrak{X})$ to $S(\mathfrak{X})$ is not only faithful but invertible. Now we will state some necessary conditions on $\mathfrak{X}$ for π^I to be invertible.

DEFINITION. The space X is *locally weakly path connected* if for each $x_0 \in X$ and each neighborhood U containing x_0 , there is an open set V , $x_0 \in V \subseteq U$, such that for each $x \in V$, there is a path in U connecting x and x_0 .

THEOREM 4. *If X is a Hausdorff, 1st countable, space such that every path map is continuous, then X is locally weakly path connected.*

PROOF. If X is not locally weakly path connected, then there is an $x_0 \in X$ and a neighborhood U containing x_0 , but *no* open set V as in the definition above. Suppose $\{U_n\}$ is a neighborhood basis at x_0 with $U \supseteq U_1 \supseteq U_2 \supseteq \dots$. We may choose $x_n \in U_n$ so that we have a sequence $x_n \rightarrow x_0$ with the property that no x_n can be connected to x_0 by a path in U . Using this sequence $\{x_n\}$ we now define, for each open $U \subseteq X$, a new set U' .

$$\begin{aligned} U' &= U && \text{if } x_0 \notin U \\ &= U \sim \{x_1, x_2, x_3, \dots\} && \text{if } x_0 \in U. \end{aligned}$$

Let X' be the space X with the topology generated by $\{U'\}$. We claim that the identity map $i: X \rightarrow X'$ is a path map. To see this, we note that the local topology at $\bar{x}$ is the same in X and X' if $\bar{x} \neq x_0$ because X is Hausdorff. The only path σ of X that might not be a path of X' is an arc that intersects infinitely many x_i 's. But if σ is continuous, and $\sigma(t_j) = x_j$ for infinitely many j 's, then there is a subsequence $\{t_{j_k}\}$ converging to a certain t_0 . Further, $\sigma(t_{j_k}) \rightarrow \sigma(t_0)$. But $x_{j_k} \rightarrow x_0$ in X , and thus $x_0 = \sigma(t_0)$. Now if σ is a path, for any open U' about x_0 , there is an ϵ such that $\sigma^{-1}(t_0 - \epsilon, t_0 + \epsilon) \subseteq U'$. But this contradicts the selection of x_0 and U . Hence there is no such σ , and i must be a path map. But i is not continuous, contradicting the assumption that every path map $f: X \rightarrow Y$ is continuous. Therefore X must be locally weakly path connected. Q.E.D.

We conclude with a brief treatment of the dual case:

THEOREM 5. *If Y is normal and its open sets are F_σ 's, then every reverse path map $f: X \rightarrow Y$ is continuous.*

PROOF. Suppose U is an open set of Y . There is a continuous map: $\phi: Y \rightarrow I$ such that $\phi(x) = 1$ iff $x \in U$. (This follows from a strong version of Urysohn's lemma, using the hypothesis that open sets are F_σ 's.)² Thus $f^{-1}(U) = (\phi f)^{-1}([0, 1))$. But because f is a reverse path map, ϕf is continuous. Hence $f^{-1}(U)$ is open. Q.E.D.

² James Dugundji, *Topology*, Allyn and Bacon, Boston, 1966, p. 148.

THEOREM 6. *If X and Y are normal spaces and their open sets are F_σ 's, and if $f: X \rightarrow Y$ is any function, then the following are equivalent:*

- (i) *f is a homeomorphism.*
- (ii) *f^{-1} exists, and f and f^{-1} are reverse path maps.*
- (iii) *$I^Y \xrightarrow{\check{f}} I^X$ is a set theoretic isomorphism. (We define $\check{f}(\phi) = \phi f$.)*
- (iv) *$\check{f}$ is a homeomorphism in the c -o topology.*

REMARK. To pursue reverse path maps we could consider the action of π_I (i.e., $\pi_I(X) = I^X$) and dualize the results for path maps.

UNIVERSITY OF ILLINOIS