ASYMPTOTIC BEHAVIOR OF LIGHT INTERIOR FUNCTIONS DEFINED IN THE UNIT DISK¹

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Let D be the unit disk, C the unit circle. Let f be a light interior function from D into the Riemann sphere W , i.e. f is a continuous open map which does not take any continuum into a single point. It is known that f has a *factorization* $f = g \circ h$ where h is a homeomorphism of the unit disk onto either the unit disk or the finite complex plane and g is a nonconstant meromorphic function. We will be concerned with the case when the range of h is the unit disk.

A simple continuous curve $\Gamma: z(t)$ ($0 \leq t < 1$) contained in D is called a *boundary path* if $|z(t)| \rightarrow 1$ as $t \rightarrow 1$. The *end* of a boundary path Γ is the set $\bar{\Gamma} \cap C$. A boundary path Γ is an *asymptotic path* of f for the value c provided $f(z(t)) \rightarrow c$ as $t \rightarrow 1$. The set $A(f)$ is defined to be the set of all $e^{i\theta}$ for which there exists an asymptotic path of f in D with end E and $e^{i\theta} \in E$. The set $A_p(f)$ is defined to be the set of all $e^{i\theta}$ for which there exists an asymptotic path of f in D with end the point $e^{i\theta}$. For a homeomorphism h of D onto D the set $B(h)$ is defined to be the set of all $e^{i\theta}$ for which there exists an asymptotic path of h in D with end E and $e^{i\theta} \in \text{int } E$, where $\text{int } E$ is the interior of E .

The *cluster set*, $C(f, \theta)$, of f at $e^{i\theta}$ is defined to be the set of all points $w \in W$ for which there exists a sequence $\{z_n\}$ of points in D with $z_n \rightarrow e^{i\theta}$ and $f(z_n) \rightarrow w$. If S is a subset of D and $e^{i\theta} \in [\bar{S} \cap D]$, then the *cluster set*, $C_S(f, \theta)$, of f at $e^{i\theta}$ relative to S is defined to be the set of all points $w \in W$ for which there exists a sequence $\{z_n\}$ of points in S with $z_n \rightarrow e^{i\theta}$ and $f(z_n) \rightarrow w$.

THEOREM. *Let f be a light interior function with factorization $f = g \circ h$. If $A(g)$ is dense on C then $A(f) \cup B(h)$ is dense on C . Furthermore, if $A_p(g)$ and $A_p(h)$ are dense on C then $A_p(f) \cup B(h)$ is dense on C .*

PROOF. Suppose $A(f)$ is not dense on C . Let $(\phi_1, \phi_2) \subset C - A(f)$ be arbitrary and let $[\theta_1, \theta_2] \subset (\phi_1, \phi_2)$, with $0 < \theta_2 - \theta_1 < 2\pi$. Let Γ_1 and Γ_2 be Jordan arcs in D at $e^{i\theta_1}$ and $e^{i\theta_2}$, respectively, with $\Gamma_1 \cap \Gamma_2 = \{0\}$. Consider the domain Δ bounded by $\Gamma_1 \cup \Gamma_2$ and the arc $[\theta_1, \theta_2]$ of C . Then h maps Δ onto a domain R in D .

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Case i. $[\bar{R} \cap C] = C_{\Gamma_1}(h, \theta_1) \cup C_{\Gamma_2}(h, \theta_2)$.

Then there exists a point $e^{i\alpha} \in C_{\Gamma_1}(h, \theta_1) \cap C_{\Gamma_2}(h, \theta_2)$, and there exist sequences $\{z_n\}$ and $\{z'_n\}$ in Γ_1 and Γ_2 , respectively, with $h(z_n) \rightarrow e^{i\alpha}$ and $h(z'_n) \rightarrow e^{i\alpha}$. Let Λ be a Jordan arc at $e^{i\alpha}$ which passes consecutively through the points $h(z_1)$, $h(z'_1)$, $h(z_2)$, $h(z'_2)$, $\dots$. By a lemma of Collingwood and Cartwright [1, Lemma 1, p. 93], either $[\theta_1, \theta_2] \subset C_\Lambda(h^{-1}, \alpha)$ or $[\theta_2, \theta_1 + 2\pi] \subset C_\Lambda(h^{-1}, \alpha)$. Thus, either $(\theta_1, \theta_2) \subset B(h)$ or $(\theta_2, \theta_1 + 2\pi) \subset B(h)$ so that $(\phi_1, \phi_2) \cap B(h) \neq \emptyset$.

Case ii. $[\bar{R} \cap C] \supset C_{\Gamma_1}(h, \theta_1) \cup C_{\Gamma_2}(h, \theta_2)$ where the inclusion is proper.

Then $E = [\bar{R} \cap C] - [C_{\Gamma_1}(h, \theta_1) \cup C_{\Gamma_2}(h, \theta_2)]$ is a nonempty open subarc of C . Let $e^{i\alpha} \in E$ with $e^{i\alpha} \in A(g)$. Then $e^{i\alpha}$ is in the end of an asymptotic path Λ of g . But $C(h^{-1}, \alpha) \subset [\theta_1, \theta_2]$, and hence $h^{-1}(\Lambda)$ is an asymptotic path of f whose end intersects $[\theta_1, \theta_2]$. Thus $[\theta_1, \theta_2] \cap A(f) \neq \emptyset$ in violation of our assumption.

Since (ϕ_1, ϕ_2) was arbitrary it follows easily that $A(f) \cup B(h)$ is dense on C and the proof of the first part is complete. The second part of the theorem is proved similarly.

If h is a homeomorphism of $\bar{D}$ onto $\bar{D}$ then $B(h) = \emptyset$ and we obtain the following corollary.

COROLLARY. *Let f be a light interior function with factorization $f = g \circ h$ with h a homeomorphism of $\bar{D}$ onto $\bar{D}$. If $A(g)$ is dense on C then $A(f)$ is dense on C . Furthermore, if $A_p(f)$ is dense on C then $A_p(f)$ is dense on C .*

The following result shows that the dense set $A_p(f)$ need not have positive measure even if f is bounded.

EXAMPLE. *There exists a bounded light interior function $f = g \circ h$ with $A_p(g)$ and $A_p(h)$ dense on C , but $A_p(f)$ is of measure zero.*

PROOF. By a theorem of Beurling and Ahlfors [4, p. 119] for each $K > 1$ there exists a K -quasiconformal homeomorphism of $\bar{D}$ onto $\bar{D}$ such that a certain set E of linear measure 2π on C is mapped onto a set F of linear measure zero on C . By a theorem of Lohwater and Piranian [3, Theorem 4, p. 11], there exists a bounded holomorphic function g which has no radial limits at points of the set F . By a theorem of Lehto and Virtanen [2, Theorem 2, p. 53], g has no point asymptotic limits at points of F . But $A_p(f) \subset C - E$, hence $\text{meas } A_p(f) = 0$ and the proof is complete.

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