NODAL ALGEBRAS OF DIMENSION p^3

JERRY I. GOLDMAN¹

1. Introduction. The algebras discussed in this paper are members of the class K , of nodal noncommutative Jordan algebras, constructed as follows by Kokoris in [3]. For $p > 2$ a prime number, F a field of characteristic p , and $n \geq 2$, let

$$B_n = F[1, x_1, \dots, x_n]/(x_1^p, \dots, x_n^p).$$

An algebra A is in K if and only if there exist p , n , and F such that $A^+ = B_n$ where multiplication in A is defined by

$$fg = f \cdot g + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot [x_i, x_j]$$

in terms of the (dot) product in B_n , and the $[x_i, x_j] = x_i x_j - x_j x_i$ are arbitrary except for the condition that at least one of these commutators be nonsingular.

Every simple nodal noncommutative Jordan algebra is in K , but not all the algebras in K are simple [3]. Properties of K have been studied in [2], [3], [4], and in several other papers, but the ideal structure of the nonsimple algebras of K is unknown. In this paper we give a complete list of the ideals of a Lie-admissible algebra in K with three generators.

THEOREM. *If $A \in K$ is Lie-admissible and of dimension p^3 , then there exist generators x, y, z such that $A = F1 + F[x, y, z]$ (vector space direct sum) with*

$$(1) \quad \begin{aligned} [y, x] &= 1 + c \cdot x^{p-1} \cdot y^{p-1}, & c &\in A, \\ [z, x] &= y^{p-1} \cdot m(z), & m(z) &\in F[1, z], \text{ and} \\ [z, y] &= x^{p-1} \cdot n(z), & n(z) &\in F[1, z]. \end{aligned}$$

If one of $m(z)$ or $n(z)$ is nonsingular, then A is simple; otherwise, the proper ideals of A are precisely $z \cdot A, z^2 \cdot A, \dots, z^{p-1} \cdot A$.

2. Proof of the theorem.

LEMMA 1. *Let $A \in K$ have at least three generators. Then there exist generators x and y of A satisfying*

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$$(2a) \quad [y, x] = 1 + c \cdot x^{p-1} \cdot y^{p-1}, \quad c \in A;$$

further, any third generator z of A may be assumed to satisfy

$$(2b) \quad \begin{aligned} [z, x] &= y^{p-1} \cdot m, & m \in A, \\ [z, y] &= x^{p-1} \cdot n, & n \in A. \end{aligned}$$

PROOF. This lemma is basically due to Oehmke [4]; however, simplicity and Lie-admissibility of A are not needed as a careful reading of his proof (pp. 417–418) will show.

LEMMA 2. Let $A \in K$ be Lie-admissible and of dimension p^3 . Then there exist generators x, y, z of A which satisfy (1). If either $m(z)$ or $n(z)$ is nonsingular, then A is simple.

PROOF. For any $a \in A$, define the derivation $D(a)$ by $bD(a) = [b, a]$ for every $b \in A$. Since A^- is a Lie algebra we have the Jacobi identity

$$(3) \quad zD(y)D(x) - zD(x)D(y) + yD(x)D(z) = 0.$$

Using Lemma 1 and (2), substitute into (3) to obtain $x^{p-1} \cdot nD(x) - y^{p-1} \cdot mD(y) + x^{p-1} \cdot y^{p-1} \cdot cD(z) = 0$. It is clear from (2b) that $x^{p-1} \cdot y^{p-1} \cdot cD(z) = 0$, thus

$$(4) \quad x^{p-1} \cdot nD(x) = y^{p-1} \cdot mD(y).$$

Without loss of generality we can assume in (2b) that $m \in F[1, x, z]$ and $n \in F[1, y, z]$. Write

$$(5) \quad m = \sum_{i,j=0}^{p-1} \alpha_{ij} x^i \cdot z^j \quad \text{and} \quad n = \sum_{i,j=0}^{p-1} \beta_{ij} y^i \cdot z^j.$$

Using (5) and (2), one can calculate that

$$\begin{aligned} x^{p-1} \cdot nD(x) &= \sum_{j=1}^{p-1} j \beta_{0j} x^{p-1} \cdot y^{p-1} \cdot z^{j-1} \cdot m + \sum_{i=1}^{p-1} i \beta_{i0} x^{p-1} \cdot y^{i-1} + \sum_{i,j=1}^{p-1} i \beta_{ij} x^{p-1} \cdot y^{i-1} \cdot z^j \end{aligned}$$

and that

$$\begin{aligned} y^{p-1} \cdot mD(y) &= \sum_{j=1}^{p-1} j \alpha_{0j} x^{p-1} \cdot y^{p-1} \cdot z^{j-1} \cdot n - \sum_{i=1}^{p-1} i \alpha_{i0} x^{i-1} \cdot y^{p-1} - \sum_{i,j=1}^{p-1} i \alpha_{ij} x^{i-1} \cdot y^{p-1} \cdot z^j. \end{aligned}$$

Equating coefficients by (4), we conclude

$$(6) \quad \begin{aligned} \alpha_{ij} &= \beta_{ij} = 0, & 1 \leq i, j \leq p-1, \\ \alpha_{i0} &= \beta_{i0} = 0, & 1 \leq i \leq p-1, \end{aligned}$$

and

$$(7) \quad \sum_{j=1}^{p-1} j(\alpha_{0j}n - \beta_{0j}m) \cdot z^{j-1} = 0.$$

To eliminate double subscripts set $\alpha_{0j} = \alpha_j$ and $\beta_{0j} = \beta_j$ for $0 \leq j \leq p-1$. Thus, (6) implies that m and n are functions of z only:

$$(8) \quad m = m(z) = \sum_{j=0}^{p-1} \alpha_j z^j \quad \text{and} \quad n = n(z) = \sum_{j=0}^{p-1} \beta_j z^j,$$

and we have (1).

To complete the proof of Lemma 2, we assume $\alpha_0 \neq 0$. Substitute (8) into (7) and again equate coefficients of the result to find

$$(9) \quad \sum_{k=0}^s k(\alpha_{s-k}\beta_k - \beta_{s-k}\alpha_k) = 0 \quad \text{for all } s = 1, \dots, p.$$

One can use the equation (9) and induction to yield the fact that $\beta_i = (\alpha_i/\alpha_0)\beta_0$, $i = 0, \dots, p-1$, and, therefore, we have

$$(10) \quad \alpha_0 n(z) = \beta_0 m(z).$$

We can now proceed as in [4, p. 421] to normalize $m(z)$ (to the form $1 + \beta z^{p-1}$) and thus conclude from Theorem 1 of [2] that A is simple. This proves Lemma 2.

REMARK. We can actually use (9) further to conclude $m(z)$ and $n(z)$ are proportional independent of whether or not one of them is nonsingular.

LEMMA 3. Let $A \in K$ be Lie-admissible and of dimension p^3 . Thus, we can assume existence of generators x, y , and z satisfying (1). If $m(z) \equiv 0(z \cdot A)$ and $n(z) \equiv 0(z \cdot A)$, then all $z^i \cdot A$, $i = 1, \dots, p-1$, are proper ideals of A .

PROOF. We have $[z^i \cdot f, g] = z^i \cdot [f, g] + i f \cdot z^{i-1} \cdot [z, g]$. So, $z^i \cdot A$ is an ideal of A^- . Since $z^i \cdot A$ is also an ideal of A^+ , the conclusion follows.

It is apparent that Lemma 2 supplies the proof of the theorem apart from the case when $m(z)$ and $n(z)$ are both singular. Consequently, we now assume that $m(z) \equiv 0(z \cdot A)$ and $n(z) \equiv 0(z \cdot A)$ and proceed to prove that the ideals of Lemma 3 are an exhaustive list.

Completion of the proof of the theorem. Let $I \neq 0$ be an ideal of A . By [1], $x^{p-1} \cdot y^{p-1} \cdot z^{p-1} \in I$. Apply (1) to calculate:

$$(x^{p-1} \cdot y^{p-1} \cdot z^{p-1}) D^{p-2}(y) = - (p-1)! x \cdot y^{p-1} \cdot z^{p-1}$$

and

$$(x \cdot y^{p-1} \cdot z^{p-1}) D^{p-2}(x) = (p-1)! x \cdot y \cdot z^{p-1}.$$

So, $x \cdot y \cdot z^{p-1} \in I$, which implies $x \cdot y D(x) \cdot z^{p-1} \in I$. The nonsingularity of $y D(x)$ implies $x \cdot z^{p-1} \in I$. Similarly, we find $z^{p-1} \in I$. Therefore, $z^{p-1} \cdot A$ is the unique minimal ideal of A , contained in every other nonzero ideal of A .

We write $A = A_1$ and define A_2 by

$$A_2 = A_1 / (z^{p-1} \cdot A_1).$$

Let λ_2 be the natural homomorphism of A_1 onto A_2 . The elements of A_2 are of the form $\sum \alpha (x\lambda_2)^i \cdot (y\lambda_2)^j \cdot (z\lambda_2)^k$ with $0 \leq i, j \leq p-1$ and $0 \leq k \leq p-2$ where we have again used dots to indicate product in A_2^+ in order to simplify the notation. A_2 , being a homomorphic image of a nodal algebra, is a nodal algebra [5]. Since A_2 inherits the basic multiplicative structure of A_1 , we can conclude as above that $(x\lambda_2)^{p-1} \cdot (y\lambda_2)^{p-1} \cdot (z\lambda_2)^{p-2}$ is in any ideal I_2 of A_2 . Thus, as before, $(z\lambda_2)^{p-2} \in I_2$ and $(z\lambda_2)^{p-2} \cdot A_2$ is the unique minimal ideal of A_2 .

As induction hypothesis assume we have constructed a sequence of nodal algebras $A_1, \dots, A_{s-1}$ for $s-1 \leq p-1$ and natural homomorphisms $\lambda_2, \dots, \lambda_{s-1}$ with λ_i mapping A_{i-1} onto A_i . Set $\pi_j = \lambda_2 \lambda_3 \dots \lambda_j$. Define A_s by

$$A_s = A_{s-1} / (z^{p-s+1} \pi_{s-1} \cdot A_{s-1}).$$

The elements of A_s are of the form $\sum \alpha (x\pi_s)^i \cdot (y\pi_s)^j \cdot (z\pi_s)^k$ where $0 \leq i, j \leq p-1$, $0 \leq k \leq p-s$ and λ_s is the natural map of A_{s-1} onto A_s . Further, we can deduce as above that $(z\pi_s)^{p-s} \cdot A_s$ is the unique minimal ideal of A_s , contained in all ideals of A_s .

Of course, the sequence $A_1, \dots, A_p$ constructed above ends with A_p which is isomorphic to the nodal algebra $F1 + F[x, y]$ where $[x, y] = 1 + \gamma x^{p-1} \cdot y^{p-1}$ and $\gamma = c(0, 0, 0)$. Thus, $A_p = A_{p-1} / (z\pi_{p-1} \cdot A_{p-1})$ is simple and $z\pi_{p-1} \cdot A_{p-1}$ is the only proper ideal of A_{p-1} . Therefore, there is only one proper ideal I_{p-2} of A_{p-2} which properly contains the minimal ideal $(z\pi_{p-2})^2 \cdot A_{p-2}$ of A_{p-2} such that

$$I_{p-2} / ((z\pi_{p-2})^2 \cdot A_{p-2}) = z\pi_{p-1} \cdot A_{p-1}.$$

Thus, $I_{p-2} = z\pi_{p-2} \cdot A_{p-2}$ and the only proper ideals of A_{p-2} are $z\pi_{p-2} \cdot A_{p-2}$ and $(z\pi_{p-2})^2 \cdot A_{p-2}$.

Now suppose by induction that the only proper ideals of A_2 are $z\pi_2 \cdot A_2$, $(z\pi_2)^2 \cdot A_2$, $\dots$, $(z\pi_2)^{p-2} \cdot A_2$. Therefore, the only proper ideals of A , which properly contain $z^{p-1} \cdot A$, must be $I_1^{(1)}$, $I_1^{(2)}$, $\dots$, $I_1^{(p-2)}$ where

$$I_1^{(j)} / (z^{p-1} \cdot A_1) = z^j \pi_2 \cdot A_2.$$

Clearly, $I_1^{(j)} = z^j \cdot A_1$ for $j = 1, \dots, p-2$. Therefore, the only proper ideals of $A_1 = A$ are $z \cdot A$, $z^2 \cdot A$, $\dots$, $z^{p-1} \cdot A$ and the theorem is proved.

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DEPAUL UNIVERSITY