SEPARABLE REPRESENTATIONS OF A W^* -ALGEBRA

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1. Introduction. The decomposition of a W^* -algebra (i.e. von Neumann algebra) as a direct integral of factors has been done for the case in which the algebra acts on a separable Hilbert space. A tempting possibility for extension of these results to some nonseparable cases would be to assume merely that the dual space M^* (as a Banach space) of a W^* -algebra M is separable in the weak* topology. The purpose of this note is to show that this possibility is vacuous by showing that any W^* -algebra M with weak* separable dual space has a weakly closed faithful * representation on a separable Hilbert space. This extends the result of Rosenthal and Arveson which makes the same assertion in the case that M is abelian [3].

2. Statement and proof of the theorem. Let M be a W^* -algebra. We use the abstract approach of Sakai [4] and consider M as a C^* -algebra which is the dual of some Banach space F . Then if M^* is the dual space of M , there is a natural imbedding of F in M^* . We shall identify F with its imbedding in M^* when convenient. It is well known that M has a weakly closed, faithful, separable representation if and only if F is norm separable.

We recall some notation from [1, p. 287]. A positive functional $f \in M^*$ is called *normal* if $f \in F$. It is called *singular* if there is no normal $g \in M^*$ with $f \geq g$. If $f \in M^*$, we may write f as $f = f^1 - f^2 + if^3 - if^4$ where $f^j \geq 0$ ($j = 1, 2, 3, 4$) and the f^j are uniquely determined by $\|f^1 - f^2\| = \|f^1\| + \|f^2\|$ and $\|f^3 - f^4\| = \|f^3\| + \|f^4\|$. We define $[f] = f^1 + f^2 + f^3 + f^4$ for any $f \in M^*$. If $f \in M^*$, $f \geq 0$, we have the unique decomposition $f = f^n + f^s$, where f^n is normal and f^s is singular and both are positive functionals.

THEOREM. *If M^* is weak* separable, then F is norm separable (and hence M has a weakly closed, faithful, separable representation).*

PROOF. Assume that $\{f_n\}_{n=1}^\infty$ is a countable weak* dense subset of M^* . Let $f = \sum_{n=1}^\infty (2^n \| [f_n] \|)^{-1} [f_n]$, convergence being assured in the norm topology of M^* . Let $f = f^n + f^s$ be the decomposition of f into normal and singular parts. Note that f is a faithful positive functional on M since $\{f_n\}_{n=1}^\infty$ is a weak* dense subset of M^* . Thus M is countably decomposable (i.e., every family of pairwise orthogonal projections is countable), so by [6] there exists an increasing sequence $\{p_n\}$

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of projections in M with $\bigvee_{n=1}^{\infty} p_n = 1$ and $f^*(p_n) = 0$ for all $n = 1, 2, \dots$. Define for each $n = 1, 2, \dots$ the spaces $F_n = \{p_n g p_n : g \in F\}$ and $M_n = \{p_n a p_n : a \in M\}$ where $p_n g p_n \in M^*$ by $(p_n g p_n)(a) = g(p_n a p_n)$ for $a \in M$. Now $F_n \subset F$ [4] for each $n = 1, 2, \dots$; so if F_n is separable for each $n = 1, 2, \dots$, F is also separable. Thus assume that n is a fixed integer for which F_n is not separable.

We remark that M_n is a W^* -algebra with predual F_n and dual $p_n M^* p_n$. We shall prove that $p_n f_k p_n \in F_n$ for each $k = 1, 2, \dots$. Since $\{p_n f_k p_n\}_{k=1}^{\infty}$ is obviously total over M_n , this would prove that F_n is weakly separable and hence norm separable. Now fix f_k and note that $f_k^i \leq \| [f_k] \| \cdot 2^k f$ for each $i = 1, 2, 3, 4$. Thus $p_n f_k^i p_n \leq \| [f_k] \| 2^k (p_n f p_n) = \| [f_k] \| 2^k (p_n f^n p_n)$, since $f^s(p_n) = 0$ implies $p_n f^s p_n = 0$ by the Schwartz inequality. Thus [5] $p_n f_k^i p_n$ is normal on M_n ($i = 1, 2, 3, 4$), so $p_n f_k p_n = p_n f_k^1 p_n - p_n f_k^2 p_n + i p_n f_k^3 p_n - i p_n f_k^4 p_n \in F_n$, for any $k = 1, 2, \dots$. Thus F_n is separable, so F is separable. Q.E.D.

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