DECOMPOSING COMPACT 3-MANIFOLDS INTO HOMEOMORPHIC HANDLEBODIES¹

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In this paper all spaces and maps considered will be in the polyhedral category in the sense of Zeeman [4]. Thus all spaces can be triangulated and all maps and subspaces will be piecewise linear (PL). This will cause no restriction as every 3-manifold has a combinatorial triangulation [1].

Suppose M is a triangulated, closed 3-manifold. It is known [3, p. 219] that if N is a regular neighborhood of the 1-skeleton of the triangulation, then N and $\text{cl}(M - N)$, the closure of its complement, are homeomorphic handlebodies. The first theorem of this paper gives a similar result for compact 3-manifolds with boundary.

THEOREM 1. *Every compact 3-manifold with nonvoid boundary contains a handlebody which is homeomorphic to the closure of its complement.*

The second result is the solution to a related question proposed by the referee. Suppose M is a closed 3-manifold and S is a surface in M separating it into compact manifolds M_1 and M_2 . We say there is a decomposition of the triple $(M; M_1, M_2)$ into homeomorphic handlebodies if there is a handlebody H in M such that $H \cong \text{cl}(M - H)$ and $M_i \cap H \cong \text{cl}(M_i - H)$ for $i = 1, 2$.

THEOREM 2. *If S is a surface (PL subspace) which separates in a closed 3-manifold M , then the triple $(M; M_1, M_2)$ defined by S has a decomposition into homeomorphic handlebodies.*

Recall that a handlebody of genus n is a space obtained by choosing $2n$ disjoint discs in the boundary of a 3-ball and identifying them in pairs by n PL homeomorphisms. Two handlebodies are homeomorphic if and only if they have the same genus and both are orientable or both are nonorientable.

In proving the two theorems some notation and a definition will be helpful. If k is a nonnegative integer, let Y_k be the set of all points (x, y) in the plane R^2 which satisfy

$$x \in \{0, 1, \dots, k\} \quad \text{and} \quad -1 \leq y \leq 1$$

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or

$$|y| = 1 \quad \text{and} \quad 0 \leq x \leq k.$$

Let $X_k = \{(x, y) \in Y_k : y \geq 0\}$. By $\text{bd } X_k$ is meant the set of points $(i, 0)$, $i = 0, 1, \dots, k$.

Now let B_k be the 3-ball in R^3 defined as the set of points (x, y, z) satisfying $-1 \leq x \leq k+1$, $0 \leq y \leq 2$, and $-1 \leq z \leq 1$. If R^2 is considered the set of all (x, y, z) in R^3 having $z = 0$, then $X_k \subset B_k$ and $X_k \cap \text{bd } B_k = \text{bd } X_k$. We call the pair (B_k, X_k) the standard k -pair.

DEFINITION. Let H be a handlebody with X a copy of X_k embedded as a PL subspace of H . X is said to be unknotted in H if there exists a ball B in H such that $B \cap \text{bd } H$ is a face F of B (see [4, Chapter III, p. 3]), $X \subset B$ with $X \cap \text{bd } B = \text{bd } X \subset F$, and (B, X) is PL homeomorphic to the standard k -pair.

Suppose that H is a handlebody of genus j and that X as above is unknotted in H . If N is a regular neighborhood of X in H which meets the boundary of H regularly (see [2, p. 722]), then it follows from the definition that $\text{cl}(H - N)$ is a handlebody of genus $j + k$.

LEMMA 1. *Let H be a handlebody contained in a closed 3-manifold M with $H \cong \text{cl}(M - H)$, and suppose Y is a copy of Y_k embedded in a one-dimensional spine Z of H . Then there is an ambient isotopy of M carrying Y onto Y' such that the sets $Y' \cap H$ and $Y' \cap \text{cl}(M - H)$ are copies of X_k which are unknotted in H and $\text{cl}(M - H)$ respectively.*

PROOF. Considering H as a 3-dimensional "thickening" of Z , there is a compact surface T which is a 2-dimensional thickening of Z and such that H is homeomorphic to $T \times [-1, 1]$ with Z corresponding to a subspace of $T \times 0$. This follows by constructing T from the handlebody presentation of H which is given by Z . If h is such a homeomorphism of $T \times [-1, 1]$ onto H , then $Z \subset h(T \times 0)$ and $h(T \times 1) \subset \text{bd } H$. Using a collar on H in M , h can be extended to an embedding, also called h , of $T \times [-1, 3]$ into M such that $h(T \times [1, 3]) \cap H = h(T \times 1)$.

Now h^{-1} gives rise to an embedding g of Y_k into $T \times 0$. Define $f: Y_k \rightarrow T \times [0, 2]$ by $f(x, y) = (g(x, y), y + 1)$. It is not difficult to see that g is ambient isotopic to f in $T \times [-1, 3]$ by an isotopy which is the identity on the boundary of $T \times [-1, 3]$. It is also evident that $f(Y_k) \cap (T \times [-1, 1])$ is unknotted in $T \times [-1, 1]$ and that $f(Y_k) \cap (T \times [1, 3])$ is unknotted in $T \times [1, 3]$. Then h induces an ambient isotopy on $h(T \times [-1, 3])$ which extends to all of M by the identity and which carries Y onto $Y' = hf(Y_k)$ as required.

PROOF OF THEOREM 1. We first consider the case that the compact 3-manifold M has connected boundary. Then $\text{bd } M$ is a closed 2-manifold and there is a handlebody G with $\text{bd } G \cong \text{bd } M$. Form the closed 3-manifold M' by attaching G to M by a homeomorphism between their boundaries. Let Y be a copy of Y_k which is embedded as a spine of G and triangulate M' so that Y is contained in the 1-skeleton.

If H_1 is a regular neighborhood of this 1-skeleton in M' , then, as remarked earlier, H_1 and $H_2 = \text{cl}(M' - H_1)$ are homeomorphic handlebodies. Applying Lemma 1, we can ambiently isotope Y to Y' such that for $i = 1, 2$, $Y' \cap H_i$ is a copy of X_k which is unknotted in H_i . Then there exists a regular neighborhood $(N; N_1, N_2)$ of Y' in $(M'; H_1, H_2)$; that is, N is a regular neighborhood of Y' in M' and each N_i is a regular neighborhood of $Y' \cap H_i$ in H_i meeting the boundary regularly. From the unknotted condition, $\text{cl}(H_1 - N_1)$ and $\text{cl}(H_2 - N_2)$ are homeomorphic handlebodies decomposing $M' - \text{int } N$. Since Y is ambiently isotopic to Y' , $M' - \text{int } N$ is homeomorphic to M , completing this case.

If $\text{bd } M$ is not connected it is a finite collection of connected, closed 2-manifolds. In this case sew a handlebody G_i into each boundary component to form the closed manifold M' . Then the above argument can be repeated for a spine Y of each G_i , completing the proof.

REMARK. Suppose that in the above proof, f is the homeomorphism of M' onto itself carrying N onto G . Then $K = f(\text{cl}(H_1 - N_1))$ is a handlebody in M which is homeomorphic to $\text{cl}(M - K)$, and in particular $K \cap \text{bd } M = \text{fr}(f(N_1))$, where fr refers to the boundary of $f(N_1)$ as a subspace of $f(H_1)$. Similar remarks apply to $\text{cl}(M - K)$.

LEMMA 2. *Suppose H_1 and H_2 are handlebodies of genus n_1 and n_2 respectively, and for $i = 1, 2$, Z_i is a copy of X_k which is unknotted in H_i . Let N_i be a regular neighborhood of Z_i in H_i meeting the boundary regularly. If $f: \text{fr } N_1 \rightarrow \text{fr } N_2$ is a PL homeomorphism, then the attachment space $\text{cl}(H_1 - N_1) \cup_f \text{cl}(H_2 - N_2)$ is a handlebody of genus $n_1 + n_2 + k$.*

PROOF. Since Z_i is unknotted in H_i , $i = 1, 2$, there is a 3-ball B_i in H_i such that (B_i, Z_i) is homeomorphic to the standard k -pair. Also we can assume that N_i is a regular neighborhood of Z_i in B_i meeting the face $B_i \cap \text{bd } H_i$ regularly. Then by [2, Corollary 1, p. 725], $\text{cl}(B_1 - N_1)$ is homeomorphic to $\text{fr } N_1 \times [0, 1]$ with $x \in \text{fr } N_1$ corresponding to $(x, 0)$. Similarly, $\text{cl}(B_2 - N_2)$ is homeomorphic to $\text{fr}(N_2) \times [-1, 0]$ with $x \in \text{fr } N_2$ corresponding to $(x, 0)$. Thus

$$\text{cl}(B_1 - N_1) \cup_f \text{cl}(B_2 - N_2) \cong \text{fr}(N_1) \times [-1, 1].$$

This is a handlebody of genus k since $\text{fr}(N_1)$ is a sphere with $k+1$ holes. Now $\text{cl}(H_i - B_i)$ is a handlebody with n_i handles and it meets B_i in a face. Thus $\text{cl}(H_1 - N_1) \cup_f \text{cl}(H_2 - N_2)$ is a handlebody of genus $n_1 + n_2 + k$.

PROOF OF THEOREM 2. We have a closed surface S separating M into compact manifolds M_1 and M_2 . Let C_i be a collar on $\text{bd } M_i = S$, $i = 1, 2$. Form $M'_1 = M_1 \cup G_1$ by sewing a handlebody G_1 to $\text{bd } M_1$ as in the proof of Theorem 1. If $f: \text{bd } M_1 \rightarrow \text{bd } M_2$ is the identity map, let $M'_2 = M_2 \cup G_2$ be the attachment space $M_2 \cup_f G_1$. Then f extends to a homeomorphism $h: G_1 \cup C_1 \rightarrow G_2 \cup C_2$. For $i = 1, 2$, triangulate M'_i so that $G_i \cup C_i$ is a subcomplex, h is a simplicial isomorphism, and there is a copy Y of Y_k which is a subcomplex of M'_1 and is a spine of G_1 .

Now carrying out the construction of the handlebody decomposition of M_1 given in Theorem 1, using $Y \subset G_1 \subset M'_1$, notice that all homeomorphisms and isotopies can be chosen to be the identity outside $G_1 \cup C_1$. Thus the homeomorphism h induces a matching decomposition of M_2 . That is, we can find K_i in M_i such that $K_i \cong \text{cl}(M_i - K_i)$ and $h(K_1 \cap \text{bd } M_1) = K_2 \cap \text{bd } M_2$. By the remark following the proof of Theorem 1, the hypotheses of Lemma 2 are satisfied with K_i corresponding to $\text{cl}(H_i - N_i)$ in that lemma. Since $h|_{\text{bd } M_1}$ is the identity, $K = K_1 \cup K_2$ is a handlebody of genus $n_1 + n_2 + k$, where n_i is determined by the 1-skeleton of the triangulation of M'_i chosen above. Similarly $\text{cl}(M - K) = \text{cl}(M_1 - K_1) \cup \text{cl}(M_2 - K_2)$ is a handlebody of genus $n_1 + n_2 + k$, so K gives the required decomposition of $(M; M_1, M_2)$.

REFERENCES

1. R. H. Bing, *An alternate proof that 3-manifolds can be triangulated*, Ann. of Math. (2) **69** (1959), 37–65. MR **20** #7269.
2. J. F. P. Hudson and E. C. Zeeman, *On regular neighborhoods*, Proc. London Math. Soc. (3) **14** (1964), 719–745. MR **29** #4063.
3. H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Teubner, Leipzig, 1934; reprint Chelsea, New York, 1947.
4. E. C. Zeeman, *Seminar on combinatorial topology*, Inst. Hautes Études Sci., Paris, 1963. (mimeographed)

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