THE RADIUS OF UNIVALENCE OF CERTAIN ANALYTIC FUNCTIONS

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1. Introduction. The writing of this paper has been motivated by a recent result of A. E. Livingston [5], and may also be regarded as a continuation of the investigation of certain classes of functions considered in [1] and [2].

Let (S) denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are regular and univalent in $E\{z: |z| < 1\}$ and which map E onto domains $D(f)$. We denote by (C) , (S^*) , and (K) the subclasses of (S) where $D(f)$ are, respectively, close-to-convex, starlike with respect to the origin, and convex. It follows that $(K) \subset (S^*) \subset (C) \subset (S)$. Let (P) denote the class of functions $p(z)$ which are regular and satisfy $p(0) = 1$, $\operatorname{Re} p(z) > 0$, for z in E . In [2] the following theorem was proven.

THEOREM A. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a member of (C) , (S^*) , or (K) . Then*

$$(1.1) \quad F(z) = (c+1)z^{-c} \int_0^z t^{c-1} f(t) dt = z + \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n} \right) a_n z^n$$

is also a member of the same class for $c = 1, 2, 3, \dots$.

Theorem A represents a generalization of the corresponding theorem by R. J. Libera [4] for the special case $c = 1$. Solving the relation (1.1) for the inverse function $f(z)$, we have

$$(1.2) \quad f(z) = \left(\frac{1}{1+c} \right) z^{1-c} [z^c F(z)]'.$$

In [5] Livingston investigates the special case of (1.2) with $c = 1$, $f(z) = \frac{1}{2} [zF(z)]'$; if $F(z) \in (S^*)$ what is the radius r_0 of starlikeness of $f(z)$? Similar questions are answered under the assumption that $F(z)$ is in (K) or in (C) . Livingston obtains the sharp result in each case to be $r_0 = \frac{1}{2}$. Furthermore, he proved that if $F(z) = z + \dots$ is regular and $\operatorname{Re}[F'(z)] > 0$ for z in E , and $f(z) = \frac{1}{2} [zF(z)]'$, then $\operatorname{Re}[f'(z)] > 0$ for $|z| < r_1 = (\sqrt{5}-1)/2$. This result is sharp.

It is the purpose of this paper to extend the results of Livingston to the general case $c = 1, 2, 3, \dots$ of the functions of (1.2). We shall

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employ the same techniques used in [5] with necessary modifications needed to treat the more general case. All results obtained will be sharp.

THEOREM 1. Let $F(z) \in (S^*)$, $f(z) = [1/(1+c)]z^{1-c}[z^c F(z)]'$, $c = 1, 2, 3, \dots$. Then $f(z)$ is starlike for $|z| < r_0$ where

$$r_0 = [-2 + (3 + c^2)^{1/2}]/(c - 1)$$

for $c > 1$, $r_0 = 1/2$ for $c = 1$. This result is sharp.

PROOF. Since $F(z) \in (S^*)$, $\operatorname{Re}[zF'(z)/F(z)] > 0$ for $|z| < 1$. Hence there exists a function $w(z)$, with $w(0) = 0$, $|w(z)| \leq |z|$, regular in $|z| < 1$ such that

$$\frac{zF'(z)}{F(z)} = \frac{z^c f(z) - c \int_0^z t^{c-1} f(t) dt}{\int_0^z t^{c-1} f(t) dt} = \frac{1 - w(z)}{1 + w(z)}$$

for $|z| < 1$. Solving for $f(z)$,

$$f(z) = \frac{(c+1) + (c-1)w(z)}{(1+w(z))z^c} \left[\int_0^z t^{c-1} f(t) dt \right].$$

A computation yields

$$\frac{zf'(z)}{f(z)} = \frac{1 - w(z) - zw'(z)}{1 + w(z)} + \frac{bw'(z)}{1 + bw(z)},$$

where $b = (c-1)/(c+1)$. Equivalently,

$$\frac{zf'(z)}{f(z)} = \frac{1 - w(z)}{1 + w(z)} - (1 - b) \left[\frac{zw'(z)}{(1 + w(z))(1 + bw(z))} \right].$$

To show that $f(z)$ is starlike in $|z| < r_0$, we must show that $\operatorname{Re}[zf'(z)/f(z)] > 0$ in $|z| < r_0$. This condition is equivalent to

$$(1.3) \quad (1 - b) \operatorname{Re} \left[\frac{zw'(z)}{(1 + w(z))(1 + bw(z))} \right] < \operatorname{Re} \left[\frac{1 - w(z)}{1 + w(z)} \right]$$

for $|z| < r_0$. On the right-side of (1.3) substitute

$$\operatorname{Re} \left[\frac{1 - w(z)}{1 + w(z)} \right] = \operatorname{Re} \left[\frac{(1 - w(z))(1 + \bar{w}(z))}{|1 + w(z)|^2} \right] = \frac{1 - |w(z)|^2}{|1 + w(z)|^2},$$

and on the left side substitute

$$\begin{aligned} \operatorname{Re} \left[\frac{zw'(z)}{(1+w(z))(1+bw(z))} \right] &= \operatorname{Re} \left[\frac{zw'(z)(1+\bar{w}(z))(1+b\bar{w}(z))}{|(1+w(z))(1+bw(z))|^2} \right] \\ &\leq \frac{|zw'(z)(1+w(z))(1+bw(z))|}{|(1+w(z))(1+bw(z))|^2} \\ &= \frac{|zw'(z)|}{|1+w(z)||1+bw(z)|}. \end{aligned}$$

Therefore the inequality (1.3) will be satisfied if

$$(1.4) \quad \frac{(1-b)|zw'(z)|}{|1+w(z)||1+bw(z)|} < \frac{1-|w(z)|^2}{|1+w(z)|^2} \quad (|z| < r_0).$$

Applying the well-known result for bounded functions

$$(1.5) \quad |w'(z)| \leq \frac{1-|w(z)|^2}{1-|z|^2} \quad (|z| < 1)$$

the inequality (1.4) will be satisfied if

$$\frac{(1-b)|z|}{|1+w(z)||1+bw(z)|} \cdot \frac{1-|w(z)|^2}{1-|z|^2} < \frac{1-|w(z)|^2}{|1+w(z)|^2}.$$

Simplifying, and writing $|z|=r$, we obtain

$$(1.6) \quad \frac{(1-b)r}{1-r^2} < \left| \frac{1+bw(z)}{1+w(z)} \right|.$$

Since $|w(z)| \leq |z|=r < 1$, $|(1+bw(z))/(1+w(z))| \geq (1+br)/(1+r)$ so that (1.6) will be satisfied if

$$\frac{(1-b)r}{1-r^2} < \frac{1+br}{1+r}.$$

This reduces to

$$1 - 2(1-b)r - br^2 > 0, \quad \text{or } (c+1) - 4r - (c-1)r^2 > 0$$

which gives the required root r_0 of Theorem 1.

To see that the result is sharp for each c , consider the function $F(z)=z/(1-z)^2$. For this function we have

$$\begin{aligned} (1.7) \quad f(z) &= \left(\frac{1}{1+c} \right) \cdot \left(\frac{z[(1+c)-(c-1)z]}{(1-z)^3} \right), \\ \frac{zf'(z)}{f(z)} &= \frac{(c+1)+4z-(c-1)z^2}{(1-z)[(c+1)-(c-1)z]}, \end{aligned}$$

so that $zf'(z)/f(z) = 0$ for $z = -r_0$. Thus $f(z)$ is not starlike in any circle $|z| < r$, if $r > r_0$. This completes the proof. Note that $\frac{1}{2} \leq r_0 \leq 1$. The value $r_0 = \frac{1}{2}$ is attained for $c = 1$ when $f(z) = \frac{1}{2}[zF(z)]'$. The value $r_0 = 1$ is attained for $c \rightarrow \infty$ when $f(z) \rightarrow F(z)$.

THEOREM 2. Let $F(z) \in (K)$, $f(z) = [1/(1+c)]z^{1-c}[z^c F(z)]'$, $c = 1, 2, 3, \dots$. Then $f(z)$ is univalent for $|z| < 1$ and is convex for $|z| < r_0$ where r_0 is the same as in Theorem 1. This result is sharp.

PROOF. We have $(1+c)f'(z) = (1+c)F'(z) + zF''(z)$. Thus

$$(1+c) \operatorname{Re} \frac{f'(z)}{F'(z)} = c + \operatorname{Re} \left[1 + \frac{zF''(z)}{F'(z)} \right] > c > 0$$

for $|z| < 1$. Hence $f(z)$ is close-to-convex (relative to $F(z)$) and therefore is univalent in $|z| < 1$. To show that $f(z)$ is convex for $|z| < r_0$, we note that

$$zf'(z) = \left(\frac{1}{1+c} \right) z^{1-c} [z^c (zF'(z))]'$$

Since $F(z) \in (K)$, $zF'(z) \in (S^*)$. Therefore, by Theorem 1, $zf'(z)$ is starlike for $|z| < r_0$ and hence $f(z)$ is convex for $|z| < r_0$. The result is sharp, for each c , for the function $F(z) = z/(1-z)$. For this function we have

$$f(z) = \frac{(1+c)z - cz^2}{(1+c)(1-z)^2},$$

$$J(z) = 1 + \frac{zf''(z)}{f'(z)} = \frac{(1+c) + 4z - (c-1)z^2}{(1-z)[(1+c) - (c-1)z]},$$

so that $J(z) = 0$ for $z = -r_0$. Thus $f(z)$ is not convex in any circle $|z| < r$, if $r > r_0$.

THEOREM 3. Let $F(z)$ be close-to-convex with respect to $G(z)$, $c = 1, 2, 3, \dots$

$$f(z) = \left(\frac{1}{1+c} \right) z^{1-c} [z^c F(z)]'; \quad g(z) = \left(\frac{1}{1+c} \right) z^{1-c} [z^c G(z)]'.$$

Then $f(z)$ is close-to-convex with respect to $g(z)$ for $|z| < r_0$ where r_0 is the same as in Theorem 1. This result is sharp.

PROOF. By definition of close-to-convexity [3], there exists a real β , $|\beta| < \pi/2$, such that $\operatorname{Re}[e^{i\beta} zF'(z)/G(z)] > 0$ for $|z| < 1$, where $G(z) \in (S^*)$. By Theorem 1, $g(z)$ is starlike for $|z| < r_0$. It is sufficient

then to prove that $\operatorname{Re} [e^{i\beta} z f'(z)/g(z)] > 0$ for $|z| < r_0$. We have

$$(1.8) \quad e^{i\beta} \left[\frac{z F'(z)}{G(z)} \right] = p(z) \cos \beta + i \sin \beta$$

where $p(z) \in (P)$. From this we obtain

$$e^{i\beta} [z^{1+c} F'(z)]' = \cos \beta [p(z)(z^c G(z))' + p'(z)(z^c G(z))] + i \sin \beta (z^c G(z))'.$$

A computation then gives

$$(1.9) \quad \begin{aligned} e^{i\beta} \left[\frac{z f'(z)}{g(z)} \right] &= e^{i\beta} \frac{[z^{1+c} F'(z)]'}{[z^c G(z)]'} \\ &= \cos \beta \left[p(z) + p'(z) \frac{z^c G(z)}{(z^c G(z))'} \right] + i \sin \beta. \end{aligned}$$

Taking real parts we have

$$(1.10) \quad \operatorname{Re} \left[e^{i\beta} \frac{z f'(z)}{g(z)} \right] \geq \cos \beta \left[\operatorname{Re} p(z) - |p'(z)| \left| \frac{z^c G(z)}{(z^c G(z))'} \right| \right].$$

If in (1.5) we substitute $w(z) = (p(z) - 1)/(p(z) + 1)$ we obtain the known result

$$(1.11) \quad |p'(z)| \leq \frac{2 \operatorname{Re} p(z)}{1 - |z|^2} \quad (|z| < 1).$$

Applying (1.11), the inequality (1.10) gives

$$(1.12) \quad \operatorname{Re} e^{i\beta} \frac{z f'(z)}{g(z)} \geq \operatorname{Re} p(z) \left[1 - \frac{2}{1 - |z|^2} \left| \frac{z^c G(z)}{(z^c G(z))'} \right| \right] \cos \beta$$

for $|z| < 1$. We note that

$$M(z) = \frac{z(z^c G(z))'}{(z^c G(z))'} = c + \frac{z G'(z)}{G(z)},$$

from which we obtain

$$|M(z)| \geq \operatorname{Re} M(z) \geq c + \frac{1 - |z|}{1 + |z|} = \frac{(c+1) + (c-1)|z|}{1 + |z|}$$

for $|z| < 1$. We then have

$$(1.13) \quad \left| \frac{z}{M(z)} \right| = \left| \frac{z^c G(z)}{(z^c G(z))'} \right| \leq \frac{|z| + |z|^2}{(c+1) + (c-1)|z|}.$$

Applying (1.13), the inequality (1.12) gives

$$\operatorname{Re} e^{i\beta} \frac{zf'(z)}{g(z)} \geq \operatorname{Re} p(z) \left[1 - \left(\frac{2}{1 - |z|^2} \right) \frac{|z| + |z|^2}{(c+1) + (c-1)|z|} \right] \cos \beta$$

for $|z| < 1$. Writing $|z| = r$ and simplifying,

$$(1.14) \quad \operatorname{Re} e^{i\beta} \frac{zf'(z)}{g(z)} \geq \operatorname{Re} p(z) \left[\frac{(c+1) - 4r - (c-1)r^2}{(1-r)[(c+1) + (c-1)r]} \right] \cos \beta.$$

The right side of (1.14) is positive provided $(c+1) - 4r - (c-1)r^2 > 0$ or $r < r_0$. The result is sharp, for each c , for the function $F(z) = G(z) = z/(1-z)^2$ which belongs to the class (S^*) and therefore to the class (C) . Moreover $zf'(z)/g(z)$ = same expression as the right side of (1.7), with the same result at $z = -r_0$.

Note that Theorem 3 may be restated without any reference to the functions $g(z)$ and $G(z)$. Let $F(z) \in (C)$, $f(z) = [1/(1+c)]z^{1-c}(z^c F(z))'$, $c \geq 1$. Then $f(z)$ is close-to-convex for $|z| < r_0$.

THEOREM 4. Let $F(z) = z + \dots$ be regular and have the property $\operatorname{Re} F'(z) > 0$ for $|z| < 1$, $f(z) = [1/(1+c)]z^{1-c}[z^c F(z)]'$, $c = 1, 2, 3, \dots$. Then $\operatorname{Re} f'(z) > 0$ for $|z| < r_1 = [-1 + (2 + 2c + c^2)^{1/2}]/(1+c)$. This result is sharp.

PROOF. Let $F'(z) = p(z)$, where $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ for $|z| < 1$. We then have

$$(1+c)f'(z) = zF''(z) + (1+c)F'(z) = (1+c)p(z) + zp'(z).$$

Applying the estimate (1.11) and writing $|z| = r$ we obtain

$$\begin{aligned} (1+c) \operatorname{Re} f'(z) &\geq (1+c) \operatorname{Re} p(z) - r |p'(z)| \\ (1.15) \quad &\geq \operatorname{Re} p(z) \left[(1+c) - \frac{2r}{1-r^2} \right] \\ &= \operatorname{Re} p(z) \left[\frac{(1+c) - 2r - (1+c)r^2}{1-r^2} \right]. \end{aligned}$$

The right side of (1.15) is positive provided $(1+c) - 2r - (1+c)r^2 > 0$, or $r < r_1$. The result is sharp, for each c , for the function $F(z) = -z - 2 \log(1-z)$. For this function we have

$$\operatorname{Re} F'(z) = \operatorname{Re} [(1+z)/(1-z)] > 0$$

for $|z| < 1$,

$$f(z) = \frac{1}{1+c} \left[\frac{(c+1)z^2 - (c-1)z}{1-z} - 2c \log(1-z) \right],$$

$$f'(z) = \frac{1}{1+c} \left[\frac{(1+c) + 2z - (1+c)z^2}{(1-z)^2} \right].$$

We then have $f'(z)=0$ when $(1+c)+2z-(1+c)z^2=0$, or $z=-r_1$. Thus $\operatorname{Re} f'(z) > 0$ in any circle $|z| < r$ if $r > r_1$.

Note that $r_1 \geq r_0$. This is to be expected, since the class of functions $F(z)$ such that $F'(z) \in (P)$ is a subclass of the class (C) . We also note that in the case where $F(z) \in (K)$, the function $f(z) = [1/(1+c)]z^{1-c}[z^c F(z)]' = [1/(1+c)][cF(z) + zF'(z)]$, $c \geq 1$, has a simple geometrical interpretation in terms of subordination. Let c_r be the convex image curve of $|z|=r < 1$ by the function $F(z) \in (K)$. Since $zF'(z)$ is an outward normal vector to c_r at the point $F(z)$, we have $F(z) \prec aF(z) + be^{i\alpha}zF'(z)$ for $a \geq 1$, $b \geq 0$, $|\alpha| \leq \pi/2$. As a special case we obtain $F(z) \prec [(1+c)/c]f(z)$.

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