RIESZ AND LAURENT DECOMPOSITIONS ON RIEMANN SURFACES

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Introduction. This paper is motivated by a paper of Heins [3] in which a classical theorem of M. Riesz was generalized to a Riemann surface setting. The theorem of Riesz in question states that if f is of class L^p on the unit circle, $1 < p < \infty$, then f admits an essentially unique decomposition of the form

$$f = f_i^* + f_e^*$$

where f_i^* is the boundary value function of a function of class H^p on the unit disc and f_e^* is the boundary value function of a function of class H^p in the complement (in the Riemann sphere) of the closed unit disc. Heins showed that if R is a compact bordered Riemann surface with double $\hat{R}$, then each $f \in L^p(\partial R)$, $1 < p < \infty$, admits a decomposition of the form

$$f = f_i^* + f_e^* + s$$

where f_i is of class H^p on R , f_e is of class H^p on $\hat{R} \setminus R$, and s is an element of a certain specified finite dimensional space of functions meromorphic on $\hat{R}$. The proof Heins gave for this result depends explicitly on the symmetry of the geometric configuration. In §II below we will obtain a generalization of this result, a generalization which requires no particular symmetry in the ambient compact surface. We will also establish a theorem which is an analogue of Laurent's theorem valid on compact Riemann surfaces. Our result contains as a special case a result of Alling [1]. §I is devoted to certain algebraic preliminaries needed in the subsequent analytic considerations.

I. Algebraic preliminaries. We begin by setting some notation. If S is a Riemann surface, let $\mathfrak{M}(S)$ and $\mathcal{O}(S)$ denote respectively the field of meromorphic functions on S and the algebra of holomorphic functions on S . If $\mathfrak{d}$ is a divisor on S , we let $\mathcal{O}(S; \mathfrak{d})$ denote the collection of elements f of $\mathfrak{M}(S)$ whose divisors (f) satisfy $(f) \geq \mathfrak{d}$. If $\mathfrak{d}$ is the unit divisor, the spaces $\mathcal{O}(S)$ and $\mathcal{O}(S; \mathfrak{d})$ coincide.

The basic tool we will use in the sequel is a simple consequence of the Riemann-Roch theorem and the following result: *Every principal*

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C-bundle on a compact Riemann surface has a nontrivial meromorphic section. Explicitly, if R is a compact Riemann surface, if $\mathfrak{U} = \{U_i\}_{i \in I}$ is an open cover for R , and if for $i, j \in I$ the functions $f_{i,j} \in \mathcal{O}(U_i \cap U_j)$ satisfy $f_{i,j} + f_{j,k} + f_{k,i} = 0$ on $U_i \cap U_j \cap U_k$, then there exist functions $f_i \in \mathfrak{M}(U_i)$ such that

$$(1.1) \quad f_{i,j} = f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j}.$$

This result is derived, e.g., in [4, pp. 658–659].

Suppose now that we have an open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of R and a base point $P \in R$. Let the genus of R be g . Assume that $f_{i,j} \in \mathcal{O}(U_i \cap U_j)$ and $f_i \in \mathfrak{M}(U_i)$ are related by (1.1). Since f_i and f_j differ on $U_i \cap U_j$ by a holomorphic function, it follows that the set Σ of points in R at which at least one of the functions f_i has a pole is finite. Moreover, if Q is a point of Σ , $Q \in U_i \cap U_j$, then the principal part of f_i at Q coincides with the principal part of f_j at Q . Suppose $\Sigma = \{Q_1, \dots, Q_\mu\}$, and for each $\nu = 1, \dots, \mu$, let p_ν be the order of the pole at Q_ν of those functions f_j for which U_j contains Q_ν . We know that p_ν is a well-defined positive integer. The Riemann-Roch theorem implies that there is an $F \in \mathfrak{M}(R)$ whose divisor is not less than the divisor $P^{-g} Q_1^{-p_1} \dots Q_\mu^{-p_\mu}$ and which has the property that the function $\tilde{f}_i = f_i - F|_{U_i}$ is holomorphic on U_i except for a possible pole, that of order no more than g , at P . We have that $\tilde{f}_i \in \mathcal{O}(U_i; P^{-g})$ and

$$(1.2) \quad f_{i,j} = \tilde{f}_i|_{U_i \cap U_j} - \tilde{f}_j|_{U_i \cap U_j}.$$

It is in this form that we will use the result. (Note that this statement admits the equivalent formulation that the cohomology group $H^1(R; \mathcal{O}_R; P^{-g}) = 0$ if $\mathcal{O}_R; P^{-g}$ is the sheaf of germs of functions whose divisors are not less than the divisor P^{-g} .)

II. The Main Results. For our first application of the material of the last section, we consider a compact Riemann surface R of genus g and in R a connected domain G with nonempty boundary Γ . We suppose Γ to consist of a finite collection of pairwise disjoint simple closed curves and that it is the boundary of $R \setminus \bar{G}$. We denote by $G_1, \dots, G_q$ the components of $R \setminus \bar{G}$, and we set $X_j = R \setminus \bar{G}_j$. Let $Q_1, \dots, Q_q$ be points of $G_1, \dots, G_q$ respectively. In this situation, we have the following analogue of Laurent's theorem:²

II.1. THEOREM. *Every $f \in \mathcal{O}(G)$ has a decomposition of the form*

$$(2.1) \quad f = f_1 + \dots + f_q + s$$

² The referee observed that in my original formulation of this theorem the domain G was insufficiently restricted.

where $f_j \in \mathcal{O}(X_j)$ and $s \in \mathcal{O}(R; Q_1^{-g} \cdots Q_q^{-g})$. If the points $Q_1, \dots, Q_q$ are non-Weierstrass points, this decomposition is unique provided we require $f_j(P) = 0$ for some fixed point $P \in G$ and $j = 1, \dots, q$.

Let us preface the proof of this theorem with a few remarks. In the case that R is the Riemann sphere and G is an annular domain, $q = 2$ and $g = 0$ so $\mathcal{O}(R; Q_1^{-g} Q_2^{-g}) = \mathbb{C}$, and we have the decomposition provided by Laurent's theorem in elementary function theory.

A less general and less precise decomposition of this sort was obtained by Alling [1]. In our notation and with the additional hypothesis that each G_j have connected boundary, Alling showed that an $f \in \mathcal{O}(G)$ can be written in the form $f = f_1 + \cdots + f_q + h$ where $f_j \in \mathcal{O}(X_j)$ and h is some function holomorphic on a neighborhood of $\bar{G}$. Plainly no uniqueness assertion is possible for this decomposition.

A point of some interest in connection with the decomposition (2.1) is the observation, already noted by Alling, that if the function f enjoys any of several restricted boundary behaviours then the functions $f_1, \dots, f_q$ will exhibit the same boundary behaviour. Thus, e.g., if f is bounded, then $f_1, \dots, f_q$ will be bounded, if f admits a continuous extension to $\bar{G}$, each f_j will extend continuously to $\bar{X}_j$, etc.

Finally, let us observe that the uniqueness assertion of our theorem cannot be obtained, in general, without the hypotheses that the points $Q_1, \dots, Q_q$ be non-Weierstrass points for R . This can be seen by very simple examples. Let R be a hyperelliptic surface of genus at least five, and let Q_1 be a Weierstrass point of R .³ Choose for G a small annulus centered around Q_1 so that $R \setminus \bar{G}$ is the union of G_2 , the "large" component, and G_1 , the "small" component, a disc containing Q_1 . Let Q_2 be any point of G_2 . The Weierstrass gap sequence at Q_1 is $1, 3, \dots, 2g - 1$. A function f meromorphic on R with a pole of order four at Q_1 and no other singularities admits at least two decompositions of the form (2.1). First of all, if we take $f_1 = f_2 = 0$ and $s = f$, we obtain a decomposition of the desired kind. Secondly, if g is a meromorphic function with only one pole, that of order two and located at Q_1 , we obtain a second decomposition of the desired form by taking $f_2 = 0$, $f_1 = -g|_{X_1}$, and $s = f + g$. (A discussion of Weierstrass points which goes far beyond what we need may be found in [2].)

We come now to the proof of the theorem.

PROOF OF THE THEOREM. Let $\mathfrak{U}^{(1)} = \{U_1^{(1)}, U_2^{(2)}\}$ be the open cover for R given by $U_1^{(1)} = G \cup \bar{G}_1$, $U_2^{(1)} = X_1$. The function f is holomorphic in $G = U_1^{(1)} \cap U_2^{(1)}$ so by §I, there exist $f_1 \in \mathcal{O}(X_1)$ and $f'_1 \in \mathcal{O}(G \cup \bar{G}_1; Q_1^{-g})$ such that $f = f_1|_G + f'_1|_G$. By the Riemann-Roch theorem, there ex-

³ Thus there exists in $\mathfrak{M}(R)$ a function with a single pole of order no more than g located at Q_1 .

ists a meromorphic function $m_1 \in \mathcal{O}(R; Q_1^{-q} Q_2^{-q})$ such that if $f_2'' = f_1' - m_1$, then $f_2'' \in \mathcal{O}(G \cup \bar{G}_1)$. We have that $f = f_1 | G + f_2'' | G + m_1 | G$. If $\mathfrak{U}^{(2)} = \{U_1^{(2)}, U_2^{(2)}\}$ is the cover given by $U_1^{(2)} = G \cup \bar{G}_1 \cup \bar{G}_2$, $U_2^{(2)} = X_2$, then by §1, $f_2'' = f_2 | (G_1 \cup G_2) + f_2' | (G_1 \cup G_2)$ where $f_2 \in \mathcal{O}(U_2^{(2)})$ and $f_2' \in \mathcal{O}(U_2^{(2)}; Q_2^{-q})$. By the Riemann-Roch theorem, there is $m_2 \in \mathcal{O}(R; Q_1^{-q} Q_2^{-q})$ such that if $f_3' = f_2' - m_2$, then $f_3'' \in \mathcal{O}(U_2^{(2)})$. This yields $f = f_1 | G + f_2 | G + f_3'' | G + (m_1 + m_2) | G$ with $f_1 \in \mathcal{O}(X_1)$, $f_2 \in \mathcal{O}(X_2)$, and $f_3'' \in \mathcal{O}(G \cup \bar{G}_1 \cup \bar{G}_2)$. If we repeat this process, we find after $q-1$ steps that

$$f = \sum_{j=1}^{q-1} f_j | G + f_q'' | G + (m_1 + \cdots + m_{q-1}) | G$$

where $f_j \in \mathcal{O}(X_j)$ and $f_q'' \in \mathcal{O}(G \cup G_1 \cup G_2 \cup \cdots \cup G_{q-1}) = \mathcal{O}(X_q)$ and $m_1 + m_2 + \cdots + m_{q-1} \in \mathcal{O}(R; Q_1^{-q} \cdots Q_q^{-q})$. Thus, we have obtained a decomposition of the desired kind.

The uniqueness assertion of the theorem may be proved as follows. We assume that the points Q_j are non-Weierstrass points and that we have decompositions

$$(2.2) \quad \begin{aligned} f &= f_1 + \cdots + f_q + s \\ &= \bar{f}_1 + \cdots + \bar{f}_q + \bar{s} \end{aligned}$$

of the form (2.1) where each f_j and $\bar{f}_j$ vanishes at the point P of G . The equations (2.2) imply that in G

$$f_1 - \bar{f}_1 = \sum_{j=2}^q (\bar{f}_j - f_j) + \bar{s} - s.$$

The left side is in $\mathcal{O}(X_1)$ and vanishes at P , and the right side is in $\mathcal{O}(G \cup \bar{G}_1; Q_1^{-q})$. As Q_1 is a non-Weierstrass point, it follows that $f_1 = \bar{f}_1$ and so that

$$f_2 - \bar{f}_2 = \sum_{j=3}^q (\bar{f}_j - f_j) + \bar{s} - s,$$

and as Q_2 is a non-Weierstrass point, $f_2 = \bar{f}_2$. Repeating this process, we find that for each j , $f_j = \bar{f}_j$ and then that $s = \bar{s}$. This concludes the proof.

The second main result we will prove is the extension to compact Riemann surfaces of the Marcel Riesz theorem mentioned in the introduction. For this purpose we consider again the compact Riemann surface R of genus g and the domains G and $G_1, \cdots, G_q$, but now it is necessary to impose additional regularity conditions on ∂G . We require that the pairwise disjoint simple closed curves which constitute Γ be analytic so that G is a regular region in the

sense of [7]. To formulate our result, it is necessary to introduce a convenient measure on Γ . For this purpose fix a base point $P_0 \in G$ and let $g(\cdot, P_0)$ be the Green's function for G with singularity at P_0 . If $\partial/\partial n$ denotes differentiation along the inner normal (with respect to G), we define the measure μ on Γ by

$$\int f d\mu = \int_{\Gamma} f(P) \frac{\partial g(P, P_0)}{\partial n} ds \quad f \in \mathcal{C}(\Gamma)$$

where ds is the differential of arclength along Γ . All measure theoretic considerations on Γ will refer to the measure μ . The measure μ depends on the choice of P_0 , but for various choices of P_0 , the resulting measures are boundedly mutually absolutely continuous so for our purposes this dependence will be irrelevant, and we will suppress it in our notation. We will write $L^p(\Gamma)$ for the L^p -space associated with μ . If Γ_j denotes ∂G_j , then $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_q$, and μ induces in an obvious way a measure on each Γ_j .

For $1 \leq p < \infty$, for D a domain in a Riemann surface, and for a divisor $\mathfrak{d} = P_1^{k_1} \dots P_m^{k_m}$ on D we define $H^p(D; \mathfrak{d})$ to be the space of those $f \in \mathcal{O}(D; \mathfrak{d})$ with the property that $|f|^p$ has a harmonic majorant on $D \setminus K$ for some and therefore every compact neighborhood K of the set $\{P_1, \dots, P_m\}$. The theory of these spaces has been developed in detail, at least for domains D with smooth boundary; it is set forth, e.g., in [3] and [6]. From this theory we will need the fact that if D is one of the domains G or G_j and if $f \in H^p(D; P^k)$, then for almost every point $Q \in \partial D$, the nontangential limit $f^*(Q) = \lim_{Q' \rightarrow Q} f(Q')$ exists and the function f^* defined almost everywhere on ∂D in this way is of class L^p . The functional

$$f \mapsto \left(\int_{\partial D} |f^*|^p d\mu \right)^{1/p}$$

is a norm on $H^p(D; P^k)$, and with respect to this norm $H^p(D; P^k)$ is a Banach space. We define $H^p(G_1 \cup \dots \cup G_q)$ to be

$$\{f \in \mathcal{O}(G_1 \cup \dots \cup G_q) : f|_{G_j} \in H^p(G_j)\}.$$

II.2. THEOREM. *If Q_0 and Q_1 are points of G and $R \setminus \bar{G}$ respectively and if $f \in L^p(\Gamma)$, $1 < p < \infty$, there exist functions $f_+ \in H^p(G)$, $f_- \in H^p(G_1 \cup \dots \cup G_q)$, and $s \in \mathcal{O}(Q_0^{-p} Q_1^{-q})$ such that almost everywhere on Γ*

$$f = f_+^* + f_-^* + s.$$

If neither Q_0 nor Q_1 is a Weierstrass point of R , this decomposition is unique provided we require that $f_+(S_0) = 0$ for some fixed point $S_0 \in G$ and $f_-(S_1) = 0$ for some $S_1 \in R \setminus \bar{G}$.

In the case that R is the Riemann sphere and G is the open unit disc, we have $g=0$ so $\mathfrak{O}(Q_0^{-g}Q_1^{-g}) = \mathbb{C}$ and the stated result is the familiar result of M. Riesz mentioned in the introduction. We will use Riesz's result in our proof; a discussion of it may be found in [5] or [8]. In the case that R is the double of G and the points Q_0 and Q_1 are conjugate under the canonical anticonformal map of R which leaves Γ fixed pointwise, the stated result is that established by Heins [3] except for an inessential difference in normalization.

PROOF. The proof consists of two steps. First we observe that using the M. Riesz theorem we can, in a sense, solve the problem locally. Having this fact, the result of §1 enables us to pass to a global solution. We obtain the uniqueness assertion by invoking a lemma proved by Royden.

Let the analytic simple closed curves which constitute Γ_k be labelled $\gamma_{k,1}, \dots, \gamma_{k,\nu(k)}$, $k=1, \dots, q$. For each $k=1, \dots, q$, and each $n=1, \dots, \nu(k)$, let $A(k, n)$ be a thin annulus about the curve $\gamma_{k,n}$, and let $A^+(k, n) = A(k, n) \cap G$, $A^-(k, n) = A(k, n) \cap G_k$. We choose the $A(k, n)$ so thin that they are pairwise disjoint. We also require that for some fixed annulus $B = \{z \in \mathbb{C}: r < |z| < R\}$, $r < 1 < R$, there exist conformal homeomorphisms $\phi_{k,n}: B \rightarrow A(k, n)$ which carry the unit circle onto $\gamma_{k,n}$, which carry $B^+ = \{z \in B: |z| < 1\}$ onto $A^+(k, n)$, and which carry $B^- = \{z \in B: |z| > 1\}$ onto $A^-(k, n)$. Let U^+ denote the open unit disc and U^- the complement in the Riemann sphere of the closed unit disc. Finally, let $L^p(T)$ be the usual L^p -space associated with Lebesgue measure on the unit circle.

The function f is in $L^p(\Gamma)$ so since $\phi_{k,n}$ is an analytic homeomorphism, the function $f \circ \phi_{k,n}$ lies in $L^p(T)$ and consequently there exist functions $f_{k,n}^+ \in H^p(U^+)$ and $f_{k,n}^- \in H^p(U^-)$ such that almost everywhere on T

$$f \circ \phi_{k,n} = (f_{k,n}^+)^* + (f_{k,n}^-)^*.$$

Carrying the functions $f_{k,n}^+$ and $f_{k,n}^-$ up to R by way of $\phi_{k,n}^{-1}$, we see that if we define A^+ by

$$A^+ = \bigcup \{A^+(k, n): k=1, \dots, q; n=1, \dots, \nu(k)\}$$

and $A_k^-, k=1, \dots, q$ by

$$A_k^- = \bigcup \{A^-(k, n): n=1, \dots, \nu(k)\},$$

then there exist functions $f_0^+ \in H^p(A^+)$ and $f_k \in H^p(A_k^-)$ such that, almost everywhere on Γ_k ,

$$f = f_0^* + f_k^*.$$

Thus we have solved the problem locally; it remains to pass to the global solution by the cohomological tool of §I. For this step, it is convenient to assume that the indexing is such that $Q_1 \in G_1$.

Let $\mathcal{U}_k$ be the open cover $\{G_k, V_k\}$ for R where

$$V_k = X_k \cup \bigcup \{A(k, n): n = 1, \dots, \nu(k)\}.$$

We have $G_k \cap V_k = A_k^-$ so $f_k \in \mathcal{O}(G_k \cap V_k)$ and it follows that for some choice of functions $f_k^- \in \mathcal{O}(G_k; Q_1^{-\theta})$ and $f_k^+ \in \mathcal{O}(V_k)$ if $k=1$ and $f_k^- \in \mathcal{O}(G_k)$, $f_k^+ \in \mathcal{O}(V_k; Q_1^{-\theta})$ if $k>1$, we have that on A_k^-

$$f_k = \bar{f}_k - f_k^+.$$

As $f_k \in H^p(A_k^-)$ and f_k^+ is holomorphic on a neighborhood of Γ_k , we see that $f_k^- \in H^p(G_k)$, ($k>1$), and $f_1^- \in H^p(G_1; Q_1^{-\theta})$. Similarly, there exist functions $f_0^+ \in H^p(G)$ and $f_0^- \in \mathcal{O}(G_1 \cup \dots \cup G_q \cup \cup A(n, k); Q_1^{-\theta})$ with the property that in A^+

$$f_0 = \bar{f}_0 - f_0^+.$$

We define functions F_+ on G and F_- on $G_1 \cup \dots \cup G_q$ by requiring that

$$F_+ = - (f_0^+ + f_1^+ + \dots + f_q^+)$$

and that on G_k

$$F_- = \bar{f}_0 + f_1^+ + \dots + f_{k-1}^+ + \bar{f}_k + f_{k+1}^+ + \dots + f_q^+.$$

The function F_+ lies in $H^p(G)$ and F_- lies in $H^p(G_1 \cup \dots \cup G_q; Q_1^{-\theta})$. On Γ_k we have almost everywhere

$$\begin{aligned} F_+^* + F_-^* &= f_0^{*-} - f_0^{+*} + f_k^{*-} - f_k^{+*} \\ &= f_0^* + f_k^* \\ &= f. \end{aligned}$$

There exists a constant m_+ and, by the Riemann-Roch theorem, an element $m_- \in \mathcal{O}(Q_0^{-\theta} Q_1^{-\theta})$ such that if $f_+ = F_+ - m_+$ and $f_- = F_- - m_-$, then f_+ is holomorphic in G and vanishes at S_0 and f_- is holomorphic in $G_1 \cup \dots \cup G_q$ and vanishes at S_1 . If we put $s = m_+ + m_-$, we have, almost everywhere on Γ

$$(2.3) \quad f = f_+^* + f_-^* + s,$$

a decomposition of the desired kind.

To prove uniqueness, assume that in addition to (2.3), we have a decomposition

$$(2.4) \quad f = g_+^* + g_-^* + t$$

of the kind described in the theorem, and suppose that neither of Q_0 nor Q_1 is a Weierstrass point for R . We have then that

$$f_+^* - g_+^* = g_-^* - f_-^* + t - s.$$

As $g_- - f_- + t - s$ belongs to the class $H^p(G_1 \cup \dots \cup G_q; Q_1^{-q})$ and $f_+ - g_+$ belongs to $H^p(G)$, a result of Royden [6, Lemma 7] yields the fact that these functions are analytic continuations of each other across Γ . As Q_1 is a non-Weierstrass point, both sides must be constant, and since $f_+(S_0) = 0 = g_+(S_0)$, we conclude that $f_+ = g_+$ and then that on Γ

$$f_-^* - g_-^* = t - s.$$

Again, this yields the fact that $f_- = g_-$ and hence that $s = t$. This concludes the proof.

We conclude with a remark on technique. We have chosen to employ the result on principal $\mathbf{C}$ -bundles quoted in §I. We could obtain both our results by invoking rather the theory of the Cauchy integral on compact Riemann surfaces as developed in [4], but this approach would be technically cumbersome at various points along the way. Of course it will be recognized that the Cauchy kernel has been used indirectly: Röhrl uses it in [4] to establish the existence of sections of principal $\mathbf{C}$ -bundles.

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