SOME REGULARITY CONDITIONS FOR THE (f, d_n, z_1) SUMMABILITY METHOD

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ABSTRACT. Regularity conditions for the generalized Lototsky summability method (f, d_n, z_1) are derived which do not restrict the Taylor coefficients of f . Some examples are given.

1. Introduction. In [4] Lototsky defined a summability method of some importance which was subsequently generalized by Jakimovski in [3] to his (F, d_n) method, and this in turn was generalized by Smith in [5] to the method (f, d_n) . Finally, Smith generalized (f, d_n) to (f, d_n, z_1) in [6]. All of the sufficiency conditions for regularity given by Smith require that the power series expansion of f about the origin have real nonnegative coefficients. In this paper we give some sufficiency conditions which do not so require.

2. Definitions and preliminaries.

DEFINITION 2.1. Let f be a nonconstant function holomorphic at the origin and let $\{d_n\}_1^\infty$ be a sequence of complex numbers with $f(z_1) + d_n \neq 0$, where z_1 is in the disk on which f is holomorphic. Let

$$\prod_1^n (f(z) + d_k) = \sum_{k=0}^{\infty} p_{nk} z^k, \quad n \geq 1.$$

Then the method (f, d_n, z_1) is defined by the matrix (a_{nk}) , where

$$\begin{aligned} a_{0k} &= 1, \quad k = 0 \\ &= 0, \quad k > 0 \end{aligned} \quad \text{and} \quad a_{nk} = \frac{p_{nk} z_1^k}{\prod_1^n (f(z_1) + d_j)}, \quad n \geq 1.$$

In terms of the above definition the Lototsky, (F, d_n) , and (f, d_n) methods are, respectively, the methods $(z, n-1, 1)$, $(z, d_n, 1)$, and $(f, d_n, 1)$. It is easy to see that if $z_1 \neq 0$ and $g(z) = f(z_1 z)$, then the methods (f, d_n, z_1) and (g, d_n) are the same.

It is well known that a matrix $A = (a_{nk})$ is regular if and only if the following conditions are met:

$$(2.2) \quad \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty,$$

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$$(2.3) \quad \lim_n \sum_{k=0}^{\infty} a_{nk} = 1,$$

$$(2.4) \quad \lim_n a_{nk} = 0, \quad k \geq 0.$$

Definition 2.1 insures that (2.3) is always satisfied, so we need only concern ourselves with the remaining criteria.

For convenience we will use the notation

$$\prod_1^n (b + d_k) = (b + d_n)!,$$

and we will set $d_n = \rho_n e^{i\theta_n}$, $-\pi < \theta_n \leq \pi$.

3. Regularity conditions for (f, d_n, z_1) .

THEOREM 3.1. *Suppose (3.2) f is holomorphic on the disk $|z| < \lambda$, $0 < |z_1| = R < \lambda$, (3.3) $|f(z_1)| = r$ and $|f(0)| = \rho < r$, and (3.4) $\{f(z)\}^n = \sum_{k=0}^{\infty} c_{nk} z^k$ with $\sum_{k=0}^{\infty} |c_{nk}| R^k \leq M r^n$, $n \geq 1$. Then (f, d_n, z_1) is regular if $(F, d_n/f(z_1))$ is regular.*

PROOF. Let $(z + d_n)! = \sum_{k=0}^n b_{nk} z^k$. The entries in the $(F, d_n/f(z_1))$ matrix are the coefficients of the polynomial

$$(3.5) \quad \prod_1^n \frac{z + d_j/f(z_1)}{1 + d_j/f(z_1)} = \frac{(f(z_1)z + d_n)!}{(f(z_1) + d_n)!} = \sum_{k=0}^n \frac{b_{nk} f(z_1)^k}{(f(z_1) + d_n)!} z^k.$$

Now,

$$(f(z) + d_n)! = \sum_{j=0}^n b_{nj} f(z)^j = \sum_{j=0}^n b_{nj} \sum_{k=0}^{\infty} c_{jk} z^k = \sum_{k=0}^{\infty} \left(\sum_{j=0}^n b_{nj} c_{jk} \right) z^k,$$

so from Definition 2.1 follows $p_{nk} = \sum_{j=0}^n b_{nj} c_{jk}$. Then,

$$\begin{aligned} \sum_{k=0}^{\infty} |a_{nk}| &= \sum_{k=0}^{\infty} \left| \frac{z_1^k}{(f(z_1) + d_n)!} \sum_{j=0}^n b_{nj} c_{jk} \right| \\ &\leq \frac{1}{|f(z_1) + d_n|!} \cdot \sum_{k=0}^{\infty} \sum_{j=0}^n |b_{nj} c_{jk} z_1^k| \\ &= \frac{1}{|f(z_1) + d_n|!} \sum_{j=0}^n |b_{nj}| \sum_{k=0}^{\infty} |c_{jk}| R^k \leq M \sum_{j=0}^n \left| \frac{b_{nj} f(z_1)^j}{(f(z_1) + d_n)!} \right|. \end{aligned}$$

The regularity of $(F, d_n/f(z_1))$ and (3.5) imply $\sum_{k=0}^{\infty} |a_{nk}| < H$, $n \geq 0$. From the formula

$$a_{nk} = \sum_{j=0}^n \frac{b_{nj} f(z_1)^j}{(f(z_1) + d_n)!} \frac{z_1^k c_{jk}}{f(z_1)^j}$$

it is clear that for a given k , a_{nk} is the n th term of the $(F, d_n/f(z_1))$ transform of the sequence $\{z_1^k c_{jk}/f(z_1)^j\}_{j=0}^{\infty}$. In view of the assumed regularity of $(F, d_n/f(z_1))$, to demonstrate (2.4) we need only show this to be a null sequence for each k . For $k=0$ we have

$$|c_{j0}/f(z_1)^j| = (\rho/r)^j \rightarrow 0, \quad j \rightarrow \infty.$$

For fixed $k > 0$, Newton's multinomial expansion yields

$$c_{jk} = \sum \frac{j!}{i_0! i_1! \cdots i_m!} c_{10}^{i_0} \cdots c_{1m}^{i_m},$$

$$i_0 + \cdots + i_m = j, \quad i_1 + 2i_2 + \cdots + mi_m = k.$$

Clearly, the number of m -tuples $(i_1, \cdots, i_m)$ is no greater than k^k , and $i_0 \leq j$, so the number of terms in the above sum is at most $k^k j$. If $1 \leq v \leq m$, then $i_v \leq k$, so $|c_{11}^{i_1} \cdots c_{1m}^{i_m}| \leq B$ for all choices of the exponents and of $m \leq k$. Also,

$$\frac{j!}{i_0! i_1! \cdots i_m!} \leq \frac{j!}{i_0!} \leq \frac{j!}{(j-k)!} \leq j^k, \quad j \geq k,$$

and $|c_{10}^{i_0}| = \rho^{i_0} = \rho^j / \rho^{j-i_0} \leq K\rho^j$ since $0 \leq j-i_0 \leq k$. Thus for $j \geq k$ we have the estimate

$$|c_{jk}/f(z_1)^j| \leq k^k j B K \rho^{jk} / r^j \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

We will need the following three lemmas.

LEMMA 3.5. If $z = \rho e^{i\theta}$, then

$$0 \leq \frac{1 + \rho}{|1 + z|} - 1 \leq \frac{2\rho(1 - \cos \theta)}{|1 + z|^2}.$$

PROOF.

$$\begin{aligned} 0 &\leq \frac{1 + \rho}{|1 + z|} - 1 \leq \left(\frac{1 + \rho}{|1 + z|} \right)^2 - 1 \\ &= \frac{(1 + \rho)^2 - (1 + 2\rho \cos \theta + \rho^2)}{|1 + z|^2} \\ &= \frac{2\rho(1 - \cos \theta)}{|1 + z|^2}. \end{aligned}$$

LEMMA 3.6. Suppose $d_k = \rho_k e^{i\theta_k}$ and

$$\sum_1^{\infty} \frac{\rho_k(1 - \cos \theta_k)}{|1 + d_k|^2} < \infty.$$

Then

$$1 \leq \prod_1^{\infty} \frac{1 + \rho_k}{|1 + d_k|} < \infty.$$

PROOF. This follows immediately from the above lemma.

LEMMA 3.7. Suppose at most finitely many d_k 's are 0,

$$\prod_1^{\infty} \frac{1 + \rho_k}{|1 + d_k|} < \infty, \quad \text{and} \quad \sum_1^{\infty} \frac{1}{|1 + d_k|} = \infty.$$

Then

$$\lim_{n \rightarrow \infty} \prod_1^n \frac{d_k}{1 + d_k} = 0,$$

the prime indicating that the product is over all nonzero factors.

PROOF. First, observe that

$$\sum_1^{\infty} \left(\frac{1 + \rho_k}{|1 + d_k|} - 1 \right) < \infty.$$

Now, let the set of positive integers be the union of the sets

$$P = \{n \mid 1 - \rho_n / |1 + d_n| > 0\} \quad \text{and} \quad Q = \{\text{all other positive integers}\}.$$

If $k \in Q$, then

$$\frac{1}{|1 + d_k|} \leq \frac{1 + \rho_k}{|1 + d_k|} - 1,$$

whence

$$(3.8) \quad \sum_{k \in Q} |1 + d_k|^{-1} < \infty.$$

It follows that $\sum_{k \in P} |1 + d_k|^{-1} = \infty$, and thus

$$\begin{aligned} \sum_{k \in P} \left\{ \frac{1}{|1 + d_k|} - \left(\frac{1 + \rho_k}{|1 + d_k|} - 1 \right) \right\} \\ = \sum_{k \in P} \left(1 - \frac{\rho_k}{|1 + d_k|} \right) = \infty, \end{aligned}$$

which in turn implies that the product

$$(3.9) \quad \prod_{k \in P}' \frac{\rho_k}{|1 + d_k|} \text{ diverges to } 0, \quad \text{i.e., } \lim_{n \rightarrow \infty} \prod_{k \leq n, k \in P}' \frac{d_k}{1 + d_k} = 0.$$

From (3.8) it is clear that

$$\sum_{k \in Q} \left(\frac{\rho_k}{|1 + d_k|} - 1 \right) = \sum_{k \in Q} \left\{ \left(\frac{1 + \rho_k}{|1 + d_k|} - 1 \right) - \frac{1}{|1 + d_k|} \right\} < \infty,$$

so

$$(3.10) \quad \prod_{k \in Q} \frac{\rho_k}{|1 + d_k|}$$

converges. The conclusion follows easily from (3.9) and (3.10).

Lemma 3.2 and formula (3.12) of [3] together imply that if only finitely many $d_n = 0$,

$$(3.11) \quad \frac{(1 + \rho_n)!}{|1 + d_n|!} \leq K, \quad n = 1, 2, \dots,$$

and $(d_n/(1+d_n))'! \rightarrow 0$ as $n \rightarrow \infty$, then (F, d_n) is regular. Lemma 3.3 and (3.12) of [3] imply that if infinitely many $d_n = 0$, then (3.11) alone suffices for the regularity of (F, d_n) . Thus it follows from Lemma 3.7 that (3.11) and $\sum_1^\infty |1 + d_n|^{-1} = \infty$ give the regularity of (F, d_n) . From Theorem 3.1 we now get

THEOREM 3.12. *Let (3.2), (3.3), and (3.4) hold and suppose*

$$\frac{(r + \rho_n)!}{|f(z_1) + d_n|!} \leq K, \quad n \geq 1,$$

and

$$\sum_1^\infty |f(z_1) + d_n|^{-1} = \infty.$$

Then (f, d_n, z_1) is regular.

COROLLARY 3.13. *Let $f(z_1) = re^{i\phi}$ and let (3.2), (3.3), and (3.4) hold. Suppose $\sum_1^\infty |f(z_1) + d_n|^{-1} = \infty$ and*

$$\sum_1^\infty \frac{(\operatorname{Im}(d_n e^{-i\phi})^{1/2})^2}{|f(z_1) + d_n|^2} = \frac{1}{2} \sum_1^\infty \frac{\rho_n [1 - \cos(\theta_n - \phi)]}{|f(z_1) + d_n|^2} < \infty.$$

Then (f, d_n, z_1) is regular.

PROOF. By Lemma 3.6,

$$\sum_1^{\infty} \frac{\rho_n [1 - \cos(\theta_n - \phi)]}{|f(z_1) + d_n|^2} < \infty \text{ implies}$$

$$1 \leq \prod_1^{\infty} \frac{1 + \rho_n/r}{|1 + d_n/f(z_1)|} = \prod_1^{\infty} \frac{r + \rho_n}{|f(z_1) + d_n|} < \infty,$$

so Theorem 3.12 gives the result.

COROLLARY 3.14. *Let the Taylor expansion of f about the origin have real nonnegative coefficients and radius of convergence $\lambda > 1$. Suppose*

$$\sum_1^{\infty} \frac{1}{|f(1) + d_n|} = \infty \quad \text{and} \quad \sum_1^{\infty} \left(\frac{\operatorname{Im} \sqrt{d_n}}{|f(1) + d_n|} \right)^2 < \infty.$$

Then (f, d_n) is regular.

This is Theorem 2.3 in [5].

4. Examples. We can easily generate examples with nonreal Taylor coefficients. Let $\beta > 0$ and define $f(z) = \psi/(1 + i\beta z) = \psi \sum_0^{\infty} (-i\beta)^k z^k$, $\psi \neq 0$ arbitrary. Then

$$\{f(z)\}^n = \left(\frac{\psi}{1 + i\beta z} \right)^n = \sum_{k=0}^{\infty} \psi^n \binom{n+k-1}{k} (-i\beta)^k z^k = \sum_{k=0}^{\infty} c_{nk} z^k.$$

If we let

$$F(z) = \frac{1}{(1-z)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k,$$

then $\sum_{k=0}^{\infty} R^k |c_{nk}| = |\psi|^n F(R\beta)$, provided $R\beta < 1$, and this is $O(r^n)$ if $|\psi|/r \leq 1 - R\beta$. Since f has radius of convergence $1/\beta$, we may choose $z_1 = i/2\beta$ and $d_n = n\psi$ to satisfy all the requirements.

Condition (3.4) is probably the hardest to satisfy when constructing examples. In this connection, the following theorem of Bajšanski [1] and Clunie and Vermes [2] gives conditions under which $\sum_{k=0}^{\infty} |c_{nk}| = O(1)$; it is then relatively easy to meet the remaining restrictions.

THEOREM 4.1. *Let f be holomorphic on the closed unit disc and let $|f(z)| < 1$ for $|z| = 1$ except at finitely many points ζ at which $|f(\zeta)| = 1$. Then, if $\{f(z)\}^n = \sum_{k=0}^{\infty} c_{nk} z^k$, it follows that $\sum_{k=0}^{\infty} |c_{nk}| = O(1)$ if and only if $\operatorname{Re} A_{\zeta} \neq 0$ for each such ζ , where $A_{\zeta} [i(z-1)]^{p_{\zeta}}$ is the lead term of the Taylor expansion about 1 of $h_{\zeta}(z) - z^{\alpha_{\zeta}}$, and where $h_{\zeta}(z) = f(\zeta z)/f(\zeta)$ and $\alpha_{\zeta} = h'_{\zeta}(1)$.*

In [2] it is shown that the functions below satisfy the conditions of Theorem 4.1 and have $f(1) = 1$.

$$(4.2) \quad f(z) = \frac{1}{2+i} (1 + iz + z^2),$$

$$(4.3) \quad f(z) = \frac{\omega z - 1}{\omega - 1} \exp\{\omega(z-1)\}, \quad \omega = e^{i\pi/3},$$

$$(4.4) \quad f(z) = \exp\{\eta(z-1) - \eta^2(z^2-1)\}, \quad \eta = e^{i\theta}, \cos \theta = 1/4.$$

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