ONE-PARAMETER GROUPS OF *-AUTOMORPHISMS OF II_1 VON NEUMANN ALGEBRAS

ROBERT R. KALLMAN¹

The purpose of this paper is to prove the following theorem.

THEOREM 1. *Let $\mathcal{R}$ be a II_1 von Neumann algebra on the Hilbert space H . Let $s \rightarrow \phi(s)$ be a one-parameter group of *-automorphisms of $\mathcal{R}$ such that $\|\phi(s)(T) - T\| \rightarrow 0$ ($s \rightarrow 0$) for all T in $\mathcal{R}$. Then $\|\phi(s) - 1\| \rightarrow 0$ ($s \rightarrow 0$).*

This theorem answers in the affirmative the question posed in Remark 3.5 of [2].

Our notation usually coincides with that of Dixmier [1]. The only exceptions are the following. If $\mathcal{R}$ is a von Neumann algebra, we denote by $P(\mathcal{R})$ the lattice of projections of $\mathcal{R}$ and by $\mathcal{R}_1$ the unit ball of $\mathcal{R}$.

Note that since $\mathcal{R}$ is a II_1 von Neumann algebra, $\mathcal{R} = \bigoplus_{\alpha \in A} \mathcal{R}_\alpha$, where each $\mathcal{R}_\alpha$ is a II_1 von Neumann algebra with a faithful finite normal trace. By Lemma 3.1 of [2], each $\phi(s)$ leaves $\text{Cent } \mathcal{R}$ pointwise fixed. Hence, each $\phi(s)$ maps each $\mathcal{R}_\alpha$ into itself. Therefore, by Lemma 3.2 of [2], it suffices to prove Theorem 1 under the assumption that $\mathcal{R}$ has a faithful finite normal trace, which we denote by $\text{Tr}(\cdot)$. In what follows, we let $\|\cdot\|_2$ denote the corresponding trace norm; i.e., if T is in $\mathcal{R}$, then $\|T\|_2 = (\text{Tr}(T^*T))^{1/2}$.

A portion of the proof of the following lemma is similar to a part of the proof of Theorem 3.4 in [2]. However, we include all details for the sake of completeness.

LEMMA 2. *Given $\epsilon > 0$, there exists $\delta > 0$ and a positive integer m such that if $P \in P(\mathcal{R})$, $S \in \mathcal{R}_1$, $\|P\|_2 \leq \delta$, then*

$$\sup_{|s| \leq 1/m} \|\phi(s)(PSP) - PSP\| \leq \epsilon.$$

PROOF. Let $M_k = [T \in \mathcal{R}_1 \mid \sup_{|s| \leq 1/k} \|\phi(s)(T) - T\| \leq \epsilon]$. Each M_k is closed in the strong operator topology, for $s \rightarrow \phi(s)(T)x$ is continuous for each $T \in \mathcal{R}$ and $x \in H$, and

$$M_k = \bigcap_{x \in H; \|x\|=1; |s| \leq 1/k} [T \in \mathcal{R}_1 \mid \|[\phi(s)(T) - T]x\| \leq \epsilon].$$

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Now R_1 is a complete metric space in the norm defined by the trace. The topology defined on R_1 by the trace norm coincides with the strong operator topology (Dixmier [1, Proposition 4, p. 62]). Hence, the Baire Category Theorem implies that there exist T_0 in R_1 , $\delta > 0$, and some positive integer m such that $\|T\| \leq 1$, $\|T - T_0\|_2 \leq \delta$ implies $\sup_{|s| \leq 1/m} \|\phi(s)(T) - T\| \leq \epsilon$.

Let $P \in P(R)$ be any projection such that $\|P\|_2 \leq \delta/4$, and let S be an arbitrary element of R_1 . Then $\|PSP + (I - P)T_0(I - P)\| \leq 1$ and

$$\begin{aligned} \|(PSP + (I - P)T_0(I - P)) - T_0\|_2 &\leq \|PSP\|_2 + \|PT_0\|_2 \\ &\quad + \|T_0P\|_2 + \|PT_0P\|_2 \leq 4\|P\|_2 \leq \delta. \end{aligned}$$

Hence,

$$\begin{aligned} \sup_{|s| \leq 1/m} \|\phi(s)(PSP) - PSP\| &\leq \sup_{|s| \leq 1/m} \|\phi(s)(PSP + (I - P)T_0(I - P)) \\ &\quad - (PSP + (I - P)T_0(I - P))\| + \sup_{|s| \leq 1/m} \|\phi(s)((I - P)T_0(I - P)) \\ &\quad - (I - P)T_0(I - P)\| \leq 2\epsilon. \quad \text{Q.E.D.} \end{aligned}$$

LEMMA 3. Let $T = \int_{-1}^1 \lambda dE(\lambda)$ be a selfadjoint element of R_1 . Let $P = E([-1, -\delta]) + E([\delta, 1])$. Suppose $\|T\|_2 \leq \delta^2$. Then $\|P\|_2 \leq \delta$.

PROOF. $\delta P \leq |T|$. Hence, $\delta\|P\|_2 \leq \| |T| \|_2 = \|T\|_2 \leq \delta^2$. Hence, $\|P\|_2 \leq \delta$. Q.E.D.

LEMMA 4. Given $\epsilon > 0$, there exists $\delta > 0$ and a positive integer m such that T in R_1 , $\|T\|_2 \leq \delta$, implies that

$$\sup_{|s| \leq 1/m} \|\phi(s)(T) - T\| \leq \epsilon.$$

PROOF. By Lemma 2, choose a positive integer m and $0 < \delta \leq \epsilon$ so that if $P \in P(R)$, $\|P\|_2 \leq \delta$, and $S \in R_1$, then

$$\sup_{|s| \leq 1/m} \|\phi(s)(PSP) - PSP\| \leq \epsilon.$$

First suppose T is selfadjoint and that $\|T\|_2 \leq \delta^2$. Let P be as in Lemma 3. Then $\|P\|_2 \leq \delta$. Hence,

$$\begin{aligned} \sup_{|s| \leq 1/m} \|\phi(s)(T) - T\| &\leq \sup_{|s| \leq 1/m} \|\phi(s)((I - P)T) - (I - P)T\| \\ &\quad + \sup_{|s| \leq 1/m} \|\phi(s)(PT) - PT\| \leq 2\|(I - P)T\| + \epsilon \leq 2\delta + \epsilon \leq 3\epsilon. \end{aligned}$$

Now if T is an arbitrary element of R_1 , then $T = T_1 + iT_2$, where T_1 and T_2 are selfadjoint elements of R_1 . Since $\|T_1\|_2 \leq \|T\|_2$ and $\|T_2\|_2 \leq \|T\|_2$, the result follows. Q.E.D.

LEMMA 5. Let A be a maximal abelian subalgebra of R . Then there exist n mutually orthogonal projections $E_1, \dots, E_n$ in A such that $\text{Tr}(E_1) = \dots = \text{Tr}(E_n) = 1/n$.

PROOF. Note that A has no minimal projections since R has no minimal projections and A is maximal abelian in R . By an easy induction argument, it suffices to prove that there exists one E in $P(A)$ such that $\text{Tr}(E) = 1/n$. Choose a maximal chain $[P_\alpha]$ of projections in A such that $\alpha \leq \beta$ implies that $P_\alpha \leq P_\beta$ and $\text{Tr}(P_\alpha) \leq 1/n$. Let $Q = V_\alpha P_\alpha$. Note that $\text{Tr}(Q) \leq 1/n$. We claim that $\text{Tr}(Q) = 1/n$. If not, choose a large positive integer m such that $2^{-m} \leq 1/n - \text{Tr}(Q)$. $I - Q = Q_1 + \dots + Q_{2^m}$, where $Q_1, \dots, Q_{2^m} \in P(A)$ are nonzero and mutually orthogonal. $Q_1, \dots, Q_{2^m}$ exist since A has no minimal projections. Since $\text{Tr}(I - Q) \leq 1$, $\text{Tr}(Q_j) \leq 2^{-m}$ for some j . Then $Q_j \perp Q$ and $\text{Tr}(Q_j + Q) \leq 1/n$, contradicting the maximality of the chain $[P_\alpha]$. Hence, $\text{Tr}(Q) = 1/n$. Q.E.D.

LEMMA 6. Given $\epsilon > 0$, there exists a positive integer m such that $\sup_{\|s\| \leq 1/m, T \in R_1} \|\phi(s)(T) - T\|_2 \leq \epsilon$.

PROOF. Given $\epsilon > 0$, choose $\delta > 0$ and m as in Lemma 2. Let T be a selfadjoint element of R_1 . Imbed T in a maximal abelian subalgebra A of R . Choose an integer n such that $1/n \leq \delta^2$. Choose orthogonal projections $E_1, \dots, E_n$ in A as given by Lemma 5. Then $T = \sum_{1 \leq i \leq n} A_i E_i$, where $A_i = T E_i$. Note that $A_i \cdot A_j = \delta_{ij} A_i^2$.

$$\begin{aligned} \phi(s)(T) - T &= \sum_{1 \leq i \leq n} \phi(s)(A_i)(\phi(s)(E_i) - E_i) \\ &\quad + \sum_{1 \leq i \leq n} (\phi(s)(A_i) - A_i) E_i. \end{aligned}$$

Let the first summation be R_1 and the second summation be R_2 .

$$\begin{aligned} \|R_1\|_2^2 &= \sum_{1 \leq i, j \leq n} \text{Tr}((\phi(s)(E_j) - E_j)\phi(s)(A_j A_i) \\ &\quad \cdot (\phi(s)(E_i) - E_i)) = \sum_{1 \leq i \leq n} \text{Tr}((\phi(s)(E_i) - E_i)^2 \phi(s)(A_i^2)) \\ &\leq \sup_{1 \leq i \leq n} \|\phi(s)(E_i) - E_i\|_2^2 \sum_{1 \leq i \leq n} \text{Tr}(\phi(s)(A_i^2)) \\ &\leq \epsilon^2 \sum_{1 \leq i \leq n} \text{Tr}(\phi(s)(A_i^2)) \end{aligned}$$

(by Lemma 2 and by the choice of the E_i)

$$\begin{aligned} &= \epsilon^2 \text{Tr}(\phi(s)(T^2)) \\ &\leq \epsilon^2. \end{aligned}$$

Similarly,

$$\begin{aligned}
 \|R_2\|_2^2 &= \sum_{1 \leq i, j \leq n} \text{Tr}((\phi(s)(A_j) - A_j)E_j E_i (\phi(s)(A_i) - A_i)) \\
 &= \sum_{1 \leq i \leq n} \text{Tr}((\phi(s)(A_i) - A_i)^2 E_i) \\
 &\leq \sup_{1 \leq i \leq n} \|\phi(s)(A_i) - A_i\|_2^2 \sum_{1 \leq i \leq n} \text{Tr}(E_i) \\
 &\leq \epsilon^2 \text{Tr}(I) \leq \epsilon^2.
 \end{aligned}$$

Since an arbitrary element of R_1 is the sum of two selfadjoint elements of R_1 ,

$$\sup_{|s| \leq 1/m; T \in R_1} \|\phi(s)(T) - T\|_2 \leq 4\epsilon. \quad \text{Q.E.D.}$$

PROOF OF THEOREM 1. Fix $\epsilon > 0$. By Lemma 4 and Lemma 6, choose a positive integer m and $\delta > 0$ such that

$$\sup_{|s| \leq 1/m; T \in R_1; \|T\|_2 \leq \delta} \|\phi(s)(T) - T\| \leq \epsilon$$

and

$$\sup_{|s| \leq 1/m; T \in R_1} \|\phi(s)(T) - T\|_2 \leq 2\delta^2.$$

Without loss of generality, we may suppose that $\delta \leq \epsilon$. Now fix a self-adjoint operator T in R_1 and fix $|s| \leq 1/m$. Let P be the projection given by Lemma 3 for the operator $\frac{1}{2}[\phi(s)(T) - T]$. Note that we have $\|P\|_2 \leq \delta$. Then $\|\phi(s)(T) - T\| \leq 2\delta + \|\phi(s)(T)P - TP\|$. By Lemma 4,

$$\begin{aligned}
 \|\phi(s)(T)P - TP\| &\leq \|\phi(s)(TP) - TP\| + \|\phi(s)(TP) - \phi(s)(T)P\| \\
 &\leq \|\phi(s)(TP) - TP\| + \|\phi(s)(P) - P\| \leq 2\epsilon.
 \end{aligned}$$

Since any element of R_1 is the sum of two elements of R_1 ,

$$\sup_{|s| \leq 1/m; T \in R_1} \|\phi(s)(T) - T\| \leq 8\epsilon. \quad \text{Q.E.D.}$$

By combining Theorem 1 with Theorem 3.4 of [2], we obtain the following result.

THEOREM 7. *Let R be a von Neumann algebra. Suppose that the I_∞ , II_∞ , and III_∞ portions of R may be expressed as a direct sum of countably decomposable von Neumann algebras. Let $\phi(s)$ be a one-parameter*

group of $*$ -automorphisms of $\mathcal{R}$ such that

$$\|\phi(s)(T) - T\| \rightarrow 0 \quad (s \rightarrow 0) \quad \text{for all } T \text{ in } \mathcal{R}.$$

Then $\|\phi(s) - 1\| \rightarrow 0 \quad (s \rightarrow 0)$.

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY