WREATH PRODUCTS AND FORMATIONS OF GROUPS

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1. Introduction. Let A and B be finite nontrivial groups and $A \text{ wr } B$ denote the wreath product of A by B . In [1] Baumslag showed that $A \text{ wr } B$ is nilpotent if and only if A and B are both p -groups for the same prime p . In [2] Durbin proved that $A \text{ wr } B$ is supersolvable if and only if

- (i) A is nilpotent,
- (ii) either B is abelian or B' is a nontrivial p -group with A a p -group for the same prime p , and
- (iii) for each prime q dividing $|A|$, $q \equiv 1 \pmod{m}$, where m is the exponent of B/Q , Q being the Sylow q -subgroup of B .

Since both the properties nilpotency and supersolvability are examples of that class of group theoretical properties called locally defined formations [3], it is natural to ask more generally for a characterization of those groups A and B such that their wreath product lies in an arbitrary locally defined formation.

The main result of this paper is

THEOREM 2. *Let $\{f_p\}$ be a collection of formations, one for each prime p and let f be the formation locally defined by the collection $\{f_p\}$. If A and B are nontrivial finite groups, then $A \text{ wr } B \in f$ if and only if the following conditions are satisfied:*

- (a) $p \nmid |A|$ and $p \nmid |B|$ if $f_p = \emptyset$,
- (b) if $f_p \neq \emptyset$
 - (i) $B/F_p(B) \in f_p$ for each $p \mid |A|$,
 - (ii) $B/O_p(B) \in f_p$ for each $p \mid |A|$ such that $F_p(A) = A$,
 - (iii) $[A/F_p(A) \text{ wr } B] \in f_p$ for each $p \mid |A|$ such that $F_p(A) < A$.

2. Notation and definitions. All groups considered are finite.

If H is a subgroup of A then $\prod(H) =$ the direct product of $|B|$ copies of $H =$ the set of all functions from B to A . $A \text{ wr } B = \prod(A) \cdot B$, the semidirect product of $\prod(A)$ by B with $\prod(A)$ normal, where $f^b(x) = f(xb^{-1})$ for $f \in \prod(A)$ and $b \in B$. A finite group H is said to be p -nilpotent if $H = PP^*$ where P^* is a normal p -complement and P is a Sylow p -group. $F_p(G) =$ the biggest p -nilpotent normal subgroup of G . $F(G) =$ the biggest nilpotent normal subgroup of G .

If U is a set of primes then $O_U(G) =$ the biggest normal U -subgroup of G .

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We will write $O_p(G)$ for $O_{\{p\}}(G)$ whenever p is a prime and p' will denote the set of primes distinct from p .

3. $F_p(A \text{ wr } B)$.

LEMMA 1. $F_p(G)$ can be characterized by

- (i) $O_{p'}(G) \subseteq F_p(G)$ and
- (ii) $O_p[G/O_{p'}(G)] = F_p(G)/O_{p'}(G)$.

PROOF. $O_{p'}(G)$ is trivially p -nilpotent since it is p -free and is a normal subgroup so $O_{p'}(G) \subseteq F_p(G)$ and in fact since $F_p(G) = PP^*$ and P^* is a characteristic subgroup we have $P^* = O_{p'}(G)$. Thus $F_p(G)/O_{p'}(G) \cong P$, hence is a p -group and is normal in $G/O_{p'}(G)$ so $F_p(G)/O_{p'}(G) \subseteq O_p[G/O_{p'}(G)]$. If the containment were proper then the complete pre-image of $O_p[G/O_{p'}(G)]$ would be a p -nilpotent normal subgroup of G which properly contains $F_p(G)$.

LEMMA 2.

$$O_U(A \text{ wr } B) = \begin{cases} \prod (A) \cdot O_U(B) & \text{if } A \text{ is a } U\text{-group,} \\ \prod O_U(A) & \text{if } A \text{ is not a } U\text{-group.} \end{cases}$$

PROOF. If A is a U -group then so is the direct product $\prod(A)$ and hence $\prod(A) \cdot O_U(B)$ is a normal U -group, therefore $\prod(A) \cdot O_U(B) \subseteq O_U(A \text{ wr } B)$.

If we denote by B_U the group $B \cap O_U(A \text{ wr } B)$ then B_U is a U -group, normal in B , so $B_U \subseteq O_U(B)$. But $O_U(A \text{ wr } B) = \prod(A) \cdot B_U \subseteq \prod(A) \cdot O_U(B)$ therefore $O_U(A \text{ wr } B) = \prod(A) \cdot O_U(B)$.

If A is not a U -group we note that in general if N is a normal subgroup of a finite group G and $O_U(G/N) = 1$ then $O_U(N) = O_U(G)$. Since

$$O_U[\prod (O_U(A))] = \prod (O_U(A))$$

it suffices to show $O_U[(A \text{ wr } B)/\prod(O_U(A))] = 1$.

Now if N is a normal subgroup of A then the group $(A/N) \text{ wr } B$ is isomorphic to the group $(A \text{ wr } B)/\prod(N)$, so in particular letting $N = O_U(A)$ we see that it suffices to show $A/O_U(A) \text{ wr } B = 1$. But the group $A^* = A/O_U(A)$ has the property that $O_U(A^*) = 1$, so we need only show that $O_U(A^*) = 1$ implies $O_U(A^* \text{ wr } B) = 1$.

Suppose $O_U(A^*) = 1$ and $O_U(A^* \text{ wr } B) \neq 1$. Since $O_U(M \times N) = O_U(M) \times O_U(N)$ and $O_U(N) = O_U(G) \cap N$ for normal subgroups M, N of G we have $1 = \prod(O_U(A^*)) = O_U \prod(A^*) = \prod(A^*) \cap O_U(A^* \text{ wr } B)$. Hence there is some $fb \in O_U(A^* \text{ wr } B)$ with $f \in \prod(A^*)$ and $1 \neq b \in B$.

Let $1 \neq a \in A^*$ be arbitrary and define $g \in \prod(A^*)$ by $g(1) = a$ and $g(x) = 1$ for all $1 \neq x \in B$. Then $[g, fb] \in \prod(A^*) \cap O_U(A^* \text{ wr } B) = 1$, in particular $[g, fb](1) = 1$. But a direct evaluation gives

$$\begin{aligned} [g, fb](1) &= (g^{-1}g^{fb})(1) = g^{-1}(1) \cdot (g^f)^b(1) \\ &= g^{-1}(1) \cdot g^f(b^{-1}) = g(1)^{-1} \cdot g(b^{-1})^f(b^{-1}) \\ &= a^{-1} \cdot 1 \neq 1, \quad \text{a contradiction.} \end{aligned}$$

Thus $O_U(A^* \text{ wr } B) = 1$.

THEOREM 1. *If A and B are nontrivial groups then*

$$\begin{aligned} F_p(A \text{ wr } B) &= \prod(A) \cdot F_p(B) && \text{if } p \nmid |A| \\ &= \prod(A) \cdot O_p(B) && \text{if } p \mid |A| \quad \text{and} \quad F_p(A) = A \\ &= \prod(F_p(A)) && \text{if } p \mid |A| \quad \text{and} \quad F_p(A) < A. \end{aligned}$$

PROOF. We use the characterization for $F_p(G)$ given in Lemma 1.

Case 1. Suppose $p \nmid |A|$. Then A is a p' -group hence $O_{p'}(A \text{ wr } B) = \prod(A) \cdot O_{p'}(B)$ by Lemma 2. Therefore $F_p(A \text{ wr } B)/O_{p'}(A \text{ wr } B) = F_p(A \text{ wr } B)/\prod(A) \cdot O_{p'}(B)$ which is isomorphic to

$$O_p[(A \text{ wr } B)/\prod(A) \cdot O_{p'}(B)]$$

by Lemma 1. This in turn is isomorphic to $O_p[B/O_{p'}(B)]$ because the groups inside the brackets $[\]$ are isomorphic. Finally $O_p[B/O_{p'}(B)]$ is equal to $F_p(B)/O_{p'}(B)$ again using the characterization for $F_p(G)$ given in Lemma 1. But the complete pre-image of $F_p(B)/O_{p'}(B)$ in $A \text{ wr } B/\prod(A) \cdot O_{p'}(B)$ under this isomorphism is

$$\prod(A) \cdot F_p(B)/\prod(A) \cdot O_{p'}(B)$$

therefore $F_p(A \text{ wr } B) = \prod(A) \cdot F_p(B)$.

CASE 2. Suppose $p \mid |A|$. Then A is not a p' -group and Lemma 2 gives $O_{p'}(A \text{ wr } B) = \prod(O_{p'}(A))$. Thus using Lemma 1

$$F_p(A \text{ wr } B)/O_{p'}(A \text{ wr } B) = O_p[(A \text{ wr } B)/\prod(O_{p'}(A))] \cong O_p(A^* \text{ wr } B)$$

where $A^* = A/O_{p'}(A)$. To apply Lemma 2 we observe that by Lemma 1 A^* is a p -group if and only if $F_p(A) = A$. Hence $F_p(A \text{ wr } B) = \prod(A) \cdot O_p(B)$ if $F_p(A) = A$. If $F_p(A) < A$, then

$$\begin{aligned} F_p(A \text{ wr } B)/O_{p'}(A \text{ wr } B) &\cong O_p(A^* \text{ wr } B) = \prod(O_p(A^*)) = \prod(F_p(A)/O_{p'}(A)), \end{aligned}$$

which implies that $F_p(A \text{ wr } B) = \prod (F_p(A))$.

The Fitting group $F(G)$ of a wreath product can be easily found using Theorem 1 and the fact that $F(G) = \bigcap F_p(G)$, the intersection taken over all primes p (p dividing $|G|$). We state without proof the

COROLLARY. *If A and B are nontrivial groups then*

$$\begin{aligned} F(A \text{ wr } B) &= \prod (A) \cdot O_p(B) && \text{if } A \text{ is a } p\text{-group,} \\ &= \prod (F(A)) && \text{if more than one prime divides } |A|. \end{aligned}$$

4. Proof of Theorem 2. By definition [3], if f is a formation locally defined by a set of formations $\{f_p\}$. Then $G \in f$ if and only if

- (a) $p \nmid |G|$ if $f_p = \emptyset$,
- (b) $G/F_p(G) \in f_p$ if $f_p \neq \emptyset$. Since $|A \text{ wr } B| = |A|^{|B|} \cdot |B|$ applying Theorem 1 we see that $A \text{ wr } B \in f$ if and only if
 - (a) $p \nmid |A|$, $p \nmid |B|$ if $f_p = \emptyset$, and if $f_p \neq \emptyset$ then
 - (b) (i)* $(A \text{ wr } B)/\prod (A) \cdot F_p(B) \in f_p$ for each $p \nmid |A|$
 - (ii)* $(A \text{ wr } B)/\prod (A) \cdot O_p \in f_p$ for each $p \mid |A|$ such that $F_p(A) = A$.
 - (iii)* $(A \text{ wr } B)/\prod (F_p(A)) \in f_p$ for each $p \mid |A|$ such that $F_p(A) < A$.

By elementary isomorphism theorems for wreath products (i)*, (ii)* and (iii)* are equivalent to (i), (ii) and (iii) respectively.

5. Applications. If it is known which wreath products are contained in the f_p which locally define f then Theorem 2 can be used to characterize those $A \text{ wr } B \in f$ in terms of conditions on A and B . Since the class of finite supersolvable groups can be described as the formation locally defined by $\{f_p\}$ where

$$\begin{aligned} f_p &= \text{the class of finite abelian groups of} \\ &\quad \text{exponent dividing } p - 1 [3, \text{Beispiel 4.8}] \end{aligned}$$

condition (b) (iii) can never occur since no wreath product of non-trivial groups is ever abelian. That is to say $F_p(A) = A$ for all p , or equivalently A is nilpotent. The characterization given by Durbin [2] is then an easy consequence of Theorem 2.

Theorem 2 yields among others characterizations of $A \text{ wr } B \in f$ in terms of conditions on A and B for the following locally defined f which are described in Huppert [4, VI §7]. We state the results without proof.

- (a) $A \text{ wr } B$ is p -solvable of p -length at most 1 if and only if
 - (i) $B/F_p(B)$ is p -free if $p \nmid |A|$,

- (ii) $B/O_p(B)$ is p -free if $p \mid |A|$ and A is p -nilpotent,
- (iii) $A/F_p(A)$ and B are p -free if $p \mid |A|$ and A is not p -nilpotent.
- (b) If $<$ is an ordering of the primes then A wr B has an ordered $<$ -Sylow tower if and only if
 - (i) $B/F_p(B)$ is divisible only by primes q with $p < q$ if $p \nmid |A|$,
 - (ii) $B/O_p(B)$ is divisible only by primes q with $p < q$ if $p \mid |A|$ and $F_p(A) = A$,
 - (iii) $A/F_p(A)$ and B are divisible only by primes q with $p < q$ if $p \mid |A|$ and $F_p(A) < A$.
- (c) A wr B is p -nilpotent if and only if
 - (i) B is p -nilpotent if $p \nmid |A|$,
 - (ii) B is a p -group and A is p -nilpotent if $p \mid |A|$.
- (d) If f is the formation of those p -solvable groups whose p -chief factors have orders $p^{n_1}, \dots, p^{n_k}$ such that $\text{l.c.m.}(n_1, \dots, n_k) = n$, then A wr $B \in f$ if and only if either
 - (i) $p \mid |A|$, A is p -nilpotent and B/P is abelian of exponent dividing $p^n - 1$ where P is a p -Sylow group of B , or
 - (ii) $p \nmid |A|$ and $B/F_p(B)$ is abelian of exponent dividing $p^n - 1$.

REFERENCES

1. G. Baumslag, *Wreath products and p -groups*, Proc. Cambridge Philos. Soc. **55** (1959), 224–231. MR **21**# 4179.
2. J. Durbin, *Finite supersolvable wreath products*, Proc. Amer. Math. Soc. **17** (1966), 215–218. MR **32**# 4192.
3. W. Gaschütz, *Zur Theorie der endlichen auflösbaren Gruppen*, Math. Z. **80** (1962/63) 300–305. MR **31**# 3505.
4. B. Huppert, *Endliche Gruppen*. I, Die Grundlehren der math. Wissenschaften, Band 134, Springer-Verlag, Berlin, 1967. MR **37**# 302.

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