ANNULUS CONJECTURE AND STABILITY OF HOMEOMORPHISMS IN INFINITE-DIMENSIONAL NORMED LINEAR SPACES

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ABSTRACT. If E is an arbitrary infinite-dimensional normed linear space, it is shown that if all homeomorphisms of E onto itself are stable, then the annulus conjecture is true for E . As a result, this confirms that the annulus conjecture for Hilbert space is true. A partial converse is that for those spaces E which have some hyperplane homeomorphic to E , if the annulus conjecture is true for E and if all homeomorphisms of E onto itself are isotopic to the identity, then all homeomorphisms of E onto itself are stable.

Let E denote a normed linear space with zero element θ , and let $B_r(x)$ be the ball in E of radius r centered at x , $S_r(x) = \text{Bd } B_r(x)$, $B_r = B_r(\theta)$, and $S_r = S_r(\theta)$. $H(E)$ and $SH(E)$ will denote the homeomorphisms on E and the stable homeomorphisms on E , respectively.

From [5] and [6] there is an inversion homeomorphism $i \in H(E)$ for E infinite-dimensional such that $i(B_1) = E - \text{Int } B_1$, $i|_{S_1} = \text{identity}$, and $i^{-1} = i$. It can be seen in [7] and [8] that this homeomorphism is a useful tool in proving topological properties of infinite-dimensional normed linear spaces. In this paper i will be used in establishing a type of engulfing theorem for infinite-dimensional E which is somewhat analogous to Stallings' Engulfing Theorem in [9]. Brown and Gluck in [3] showed that $SH(E) = H^+(E)$ implies the annulus conjecture for finite-dimensional E , where $H^+(E)$ is the group of orientation preserving homeomorphisms of E . As an application of the engulfing theorem in this paper, it will be shown that $SH(E) = H(E)$ implies the annulus conjecture for infinite-dimensional E (a slightly different technique than used in [3]). As a result, since Wong in [10] showed that $SH(l_2) = H(l_2)$, the annulus conjecture is true in Hilbert space and is consequently true in all infinite-dimensional separable Banach spaces, or more generally, in all infinite-dimensional separable Fréchet spaces (see [4] and [1]).

By a *cell* in E is meant a subset C of E such that there exists a homeomorphism from the pair (B_1, S_1) onto the pair $(C, \text{Bd } C)$. A

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closed subset K of $E - \text{Int } C$ is a *collar* of C if there exists a homeomorphism h from the pair (B_2, B_1) onto the pair $(K \cup C, C)$ such that $h(S_2) = \text{Bd } (K \cup C)$. C is *tame* if there is an $h \in H(E)$ such that $h(B_1) = C$. In [7] and [8] it is shown that C is tame if and only if it has a collar. By a *sphere* in E is meant a closed homeomorph of S_1 in E . A sphere is *tame* if it is the boundary of a tame cell. Sanderson showed in [8] that a sphere is tame if and only if it is bicollared. $A \subset E$ is an *annulus* if there exists a homeomorphism h from $B_2 - \text{Int } B_1$ onto A such that $\text{Bd } A = h(S_1 \cup S_2)$.

The annulus conjecture for E is stated in [7] as follows.

A_E : If C is a tame cell contained in $\text{Int } B_1$, then there exists a homeomorphism h of B_1 onto itself such that $h(B_{1/2}) = C$ and $h|_{S_1} = \text{identity}$.

For infinite-dimensional E , it can be seen by using i that A_E is equivalent to the statement: the region between two disjoint tame spheres is an annulus.

$h \in H(E)$ is *stable* if it can be written as a finite composition of elements of $H(E)$ each of which is the identity on some open subset of E . The following is the stability conjecture for E .

S_E : $SH(E) = H(E)$ (for E finite-dimensional $SH(E) = H^+(E)$).

LEMMA 1. If E is infinite-dimensional and C is a collared cell with collar K , then there exists a collar $K' \subset K$ of C and $g \in H(E)$ such that $g(C) = E - \text{Int } B_1$ and $g(C \cup K') = E - \text{Int } B_{1/2}$.

PROOF. Let $h: (C \cup K, \text{Bd } [C \cup K]) \rightarrow (B_3, S_3)$ be a homeomorphism such that $h(C) = B_1$. Set $K' = h^{-1}(B_2 - \text{Int } B_1)$. By Lemma 6.1 of [7], there exists $f \in H(E)$ such that $f|_{K' \cup C} = h|_{K' \cup C}$. Let $k: B_2 - \text{Int } B_1 \rightarrow B_1 - \text{Int } B_{1/2}$ be a homeomorphism such that $k(S_2) = S_{1/2}$ and $k|_{S_1} = \text{identity}$. Define the homeomorphism $f': E - \text{Int } B_1 \rightarrow B_1 - \theta$ by letting $f'|_{(B_2 - \text{Int } B_1)} = k$ and, for $E - B_2$, letting f' map $[x: \infty)$, $x \in B_2$, linearly onto $[k(x): \theta)$. Let $j: B_1 - \theta \rightarrow B_1$ be a homeomorphism such that $j|_{(B_1 - \text{Int } B_{1/2})} = \text{identity}$. Define $g' \in H(E)$ by $g'(x) = i(x)$ if $x \in B_1$, and $g'(x) = jf'(x)$ if $x \in E - B_1$. Finally define $g \in H(E)$ by $g = g'f$.

THEOREM 1 (ENGULFING THEOREM). Let E be infinite-dimensional, let $A \subset E$, let C be a collared cell in E with collar K such that $(C \cup K) \cap A = \emptyset$, and let $U \subset E$ be open such that $[E - (C \cup K)] \cap U \neq \emptyset$. Then there exists $h \in H(E)$ such that $A \subset h(U)$ and $h|_C = \text{identity}$.

PROOF. By Lemma 1, let $K' \subset K$ be a collar of C and $g \in H(E)$ be such that $g(C) = E - \text{Int } B_1$ and $g(C \cup K') = E - \text{Int } B_{1/2}$. Choose $B_\gamma(x) \subset g(U) \cap \text{Int } B_{1/2}$. Let $f \in H(E)$ be such that $B_{1/2} = f[B_\gamma(x)]$ and

$f|(E-B_1)=\text{identity}$. Define $h \in H(E)$ by $h=g^{-1}fg$.

Using a proof very similar to the proof of the above theorem, the following result concerning the fattening of a collar can be established.

THEOREM 2. *Let E be infinite-dimensional, let $A \subset E$, and let C be a collared cell in E with collar K such that $C \cup \text{Cl } A \neq E$. Then there exists $h \in H(E)$ such that $A \subset h(C \cup K)$ and $h|_C = \text{identity}$.*

PROOF. By Lemma 1, let $K' \subset K$ be a collar of C and $g \in H(E)$ be such that $g(C) = E - \text{Int } B_1$ and $g(C \cup K') = E - \text{Int } B_{1/2}$. Choose $B_\gamma(x) \subset \text{Int } B_1 - g(A)$. Let $f \in H(E)$ be such that $B_{1/2} = f[B_\gamma(x)]$ and $f|(E-B_1)=\text{identity}$. Define $h \in H(E)$ by $h=g^{-1}f^{-1}g$.

LEMMA 2. *Let A be a nondense subset of E , and let $f \in H(E)$ and $U \subset A$ be open such that $f|_U = \text{identity}$. Then there exist $f_1, f_2, f_3 \in H(E)$ and U_1, U_2, U_3 open subsets of E such that $U_1 \subset E - A$, $U_2 \subset E - f_1(A)$, $U_3 \subset E - f_2f_1(A)$, $f = f_3f_2f_1$, and $f_i|_{U_i} = \text{identity}$ for $i=1, 2, 3$.*

PROOF. Let $B_\gamma(x) \subset U$ and $B_\delta(y) \subset E - A$. Let $N > M > 0$ be such that $\text{Int } B_\delta(y) \cap \text{Int } B_M(x) \neq \emptyset$ and $\text{Int } B_\delta(y) \cap [E - B_N(x)] \neq \emptyset$. Define $f_1 \in H(E)$ such that $f_1[B_M(x)] = B_\gamma(x)$ and $f_1|[E - B_N(x)] = \text{identity}$. Define $f_2, f_3 \in H(E)$ by $f_2 = f$ and $f_3(x) = ff_1^{-1}f^{-1}(x)$ if $x \in f[B_N(x)]$, and $f_3(x) = x$ if $x \in E - f[B_N(x)]$. Choose U_1 open in $B_\delta(y) - B_N(x)$, U_2 open in $B_\gamma(x) \cap f_1[B_\delta(y)]$, and U_3 open in $f[B_\delta(y) - B_N(x)]$.

LEMMA 3. *If $f \in SH(E)$, then there exist, for $i=1, \dots, n$, $f_i \in H(E)$ and open $U_i \subset E - f_{i-1} \dots f_1 f_0(B_1)$ ($f_0 = \text{identity}$) such that $f_i|_{U_i} = \text{identity}$ and $f = f_n \dots f_1$.*

PROOF. Let $f = g_m \dots g_1$ and $V_i, i=1, \dots, m$, be open such that $g_i|_{V_i} = \text{identity}$. Let k be such that $1 \leq k \leq m$. If

$$V_k \cap [E - g_{k-1} \dots g_1 g_0(B_1)] \neq \emptyset$$

($g_0 = \text{identity}$), then choose U_{k1} open in $V_k \cap [E - g_{k-1} \dots g_1 g_0(B_1)]$ and let $f_{k1} = g_k$. If $V_k \cap [E - g_{k-1} \dots g_1 g_0(B_1)] = \emptyset$, then by Lemma 2, there exist $f_{k1}, f_{k2}, f_{k3} \in H(E)$, open $U_{k1} \subset E - g_{k-1} \dots g_1 g_0(B_1)$, open $U_{k2} \subset E - f_{k1}g_{k-1} \dots g_1 g_0(B_1)$, and open $U_{k3} \subset E - f_{k2}f_{k1}g_{k-1} \dots g_1 g_0(B_1)$ such that $g_k = f_{k3}f_{k2}f_{k1}$ and $f_{ki}|_{U_{ki}} = \text{identity}$, $i=1, 2, 3$. The f_{ij} and U_{ij} can be relabeled to give the desired f_i and $U_i, i=1, \dots, n$.

ADDENDUM TO LEMMA 3. *In Lemma 3 the f_i can further be chosen so that $f_i \dots f_1(B_1) \subset f_{i+1} \dots f_1(B_1)$ for $i=1, \dots, n-1$.*

PROOF. Let $U'_n = U_n$ and $f'_n = f_n$. For k such that $1 \leq k \leq n-1$, define $U'_{n-k}, f'_{n-k}, U''_{n-k+1}$, and f''_{n-k+1} by an inductive step as follows

Choose $B_\gamma(x) \subset f_i^{-1} \cdots f_{n-k}^{-1} f_{n-k+1}^{-1} f_{n-k} \cdots f_1(B_1)$, $B_\delta(y) \subset f_1^{-1} \cdots f_{n-k}^{-1}(U_{n-k})$, and $B_\sigma(z) \subset f_1^{-1} \cdots f_{n-k}^{-1}(U'_{n-k+1})$ such that $B_\gamma(x) \cap (B_\delta(y) \cup B_\sigma(z)) = \emptyset$. Let $g \in H(E)$ be such that $g(B_1) = B_\gamma(x)$ and $g|_{(B_\delta(y) \cup B_\sigma(z))} = \text{identity}$. Define $U'_{n-k} = f_{n-k} \cdots f_1(\text{Int } B_\delta(y))$, $f'_{n-k} = f_{n-k} \cdots f_1 g f_1^{-1} \cdots f_{n-k-1}$, $U''_{n-k+1} = f_{n-k} \cdots f_1(\text{Int } B_\sigma(z))$, and $f'_{n-k+1} = f'_{n-k+1} f_{n-k} (f'_{n-k})^{-1}$. Finally let $U'_i = U_i$ and $f'_i = f_i$. The f'_i and U'_i are then the desired homeomorphisms and open sets.

THEOREM 3. *If E is infinite-dimensional and if $f \in SH(E)$ such that $f(B_1)$ is bounded, then there exists $N > 0$ and $g \in H(E)$ such that $g|_{B_1} = f|_{B_1}$ and $g|(E - B_N) = \text{identity}$.*

PROOF. By Lemma 3 and its addendum, there exists, for $i=1, \dots, n$, $f_i \in H(E)$ and open $U_i \subset E - f_{i-1} \cdots f_1 f_0(B_1)$ ($f_0 = \text{identity}$) such that $f_i|_{U_i} = \text{identity}$, $f = f_n \cdots f_1$, and for $i=1, \dots, n-1$, $f_i \cdots f_1 f_0(B_1) \subset f_{i+1} \cdots f_1 f_0(B_1)$. Without loss of generality it can be assumed that each U_i is bounded. Choose $N > 0$ such that $B_1 \cup f(B_1) \cup [U_{i=1}^n U_i] \subset B_N$. Choose $B_\gamma(x) \subset f_1(B_1)$ and $B_\delta(y) \subset U_1$. Let $g' \in H(E)$ be such that $g'[B_\gamma(x)] = B_1$ and $g'|_{[B_\delta(y) \cup (E - B_N)]} = \text{identity}$. Redefine f_1 to be $f_1 g'$. For each $i=1, \dots, n$, by the Engulfing Theorem there exists $g_i \in H(E)$ such that $E - B_N \subset g_i(U_i)$ and $g_i|_{f_i \cdots f_1[B_\gamma(x)]} = \text{identity}$. Define $f'_i = g_i f_i g_i^{-1}$. Then define the desired g by $g = f'_n \cdots f'_1 (g')^{-1}$.

THEOREM 4. S_E implies A_E .

PROOF (FOR E INFINITE-DIMENSIONAL). Let C be a tame cell contained in $\text{Int } B_1$. By the Half-open Annulus Theorem [7], there exists a homeomorphism f of $\text{Int } B_1$ onto itself such that $f(B_{1/2}) = C$. By a slight modification of Lemma 3.1 in [7], there exists a cell $C' \subset B_1$ such that $C \cup B_{1/2} \subset C'$ and $B_1 - \text{Int } C'$ is a collar of C' . Define $g: \text{Int } B_1 \rightarrow E$ radially in the following manner. Set $S' = \text{Bd } C'$. For $x \in E$, because of the construction of C' in Lemma 3.1 in [7], $(\text{Ray } [\theta: x]) \cap S'$ is a single point—call it x_1 . Also $(\text{Ray } [\theta: x]) \cap S_1 = x_*$. Let g map $[x_1: x_*]$ linearly onto $[x_1: \infty)$ and let $g|_{C'} = \text{identity}$. Set $\tilde{f} = g f g^{-1}$. Then since S_E is true, $\tilde{f} \in SH(E)$. Hence by Theorem 3 there exist $N > 0$ and $\tilde{g} \in H(E)$ such that $\tilde{g}|_{B_1} = \tilde{f}|_{B_1}$ and $\tilde{g}|_{(E - B_N)} = \text{identity}$. Let $h: \text{Int } B_1 \rightarrow \text{Int } B_1$ be defined by $h = g^{-1} \tilde{g} g$. Finally h can be extended to the identity on S_1 .

COROLLARY 1. A_E is true in all separable infinite-dimensional Fréchet spaces.

The annulus conjecture for the Hilbert cube has been observed to be true by Anderson in Corollary 10.6 of [2]. His proof of this also

follows from the fact that homeomorphisms of the Hilbert cube onto itself are stable, which he shows in that paper.

Let $HI(E)$ denote the homeomorphisms on E which are isotopic to the identity. I_E will be the conjecture that $HI(E) = H(E)$ (use orientation preserving homeomorphisms for E finite-dimensional).

The following theorem is well known and can be found for example proved for the special spaces of E^n , s , and I^∞ in [3], [10], and [1], respectively.

THEOREM 5. S_E implies I_E .

PROOF. Let $h \in SH(E)$.

Case 1. Assume that h is fixed on B_1 . Define

$$H_t(x) = (1+t)/(1-t)h[(1-t)/(1+t)x],$$

for $0 \leq t < 1$, and $H_t(x) = x$, for $t = 1$. Let $x \in E$ and $\gamma > 0$, and choose $\delta = \max[(\|x\| + \gamma - 1)/(\|x\| + \gamma + 1), 0]$. Then for any $y \in B_\gamma(x)$ and $t \in (\delta, 1]$, $H_t(y) = y$. Thus H_t is an isotopy from h to the identity.

Case 2. $h = h_n \cdot \dots \cdot h_1$, where each h_i is fixed on some $B_{\gamma_i}(x_i)$. Then as in the above case define isotopies H_i^t from h_i to the identity. Finally define the isotopy $F_t = H_i^t \cdot \dots \cdot H_1^t$ from h to the identity.

LEMMA 4. Let A_E hold, let C be a tame cell, and let $B_\gamma(x) \subset \text{Int } C$. Then there exists $g \in H(E)$ such that $g(C) = B_{\gamma+1}(x)$ and $g|_{B_\gamma(x)} = \text{identity}$.

PROOF. Let $h \in H(E)$ be such that $h(B_1) = C$. By A_E , let $f \in H(E)$ be such that $f(B_{1/2}) = h^{-1}(B_\gamma(x))$ and $f|(E - B_1) = \text{identity}$. Define $j \in H(E)$ so that $j(B_{1/2}) = B_\gamma(x)$ and $j(B_1) = B_{\gamma+1}(x)$. Let p_r^x be the projection of $E - \theta$ onto $S_r(x)$ along the rays emanating from x . Define $k \in H(E)$ so that $k|_{B_\gamma(x)} = hfj^{-1}|_{B_\gamma(x)}$, and for $y \in S_r$, $r > \gamma$, $k(y) = p_r^x k p_\gamma^x(y)$. Then set $g = k j f^{-1} h^{-1}$.

THEOREM 6. A_E and I_E together imply S_E for all normed linear spaces E which contain a hyperplane homeomorphic to E .

PROOF. Let $h \in H(E)$, and let $B_\gamma(x) \subset h(\text{Int } B_1)$. Then by Lemma 4, there exists $g \in H(E)$ such that $gh(B_1) = B_{\gamma+1}(x)$ and $g|_{B_\gamma(x)} = \text{identity}$. Let $k \in H(E)$ be such that $k(B_{\gamma+1}(x)) = B_1$ and $k|(E - B_N) = \text{identity}$ for some N . In [6] it is seen that all hyperplanes of E are homeomorphic to S_1 in E . Then because I_E holds, there exists an isotopy $F_t: S_1 \rightarrow S_1$, $0 \leq t \leq 1$, such that $F_0 = kgh|_{S_1}$ and $F_1 = \text{identity}$. Let p_r , $r > 0$, be the radial projection of $E - \theta$ onto S_r . Then define $f \in H(E)$ as follows. For $x \in S_r$, $1 \leq r \leq 2$, let $f(x) = p_r F_{r-1} p_1(x)$, $f|_{B_1}$

$=kgh|B_1$, and $f|(E-B_2)=\text{identity}$. Then $h=g^{-1}k^{-1}f(f^{-1}kgh)$, which is stable.

COROLLARY 2. A_E and I_E together are equivalent to S_E for all normed linear spaces E which contain a hyperplane homeomorphic to E .

Similar arguments give the finite-dimensional result found in [3].

THEOREM 7. A_n and I_{n-1} together are equivalent to S_n .

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